Announcements

- HW0
- Office hours
- Instant questions
- Demos
- CBTF reservations

Today

- Errors
- Conditioning
- FP

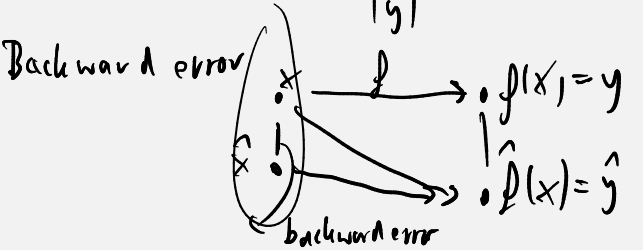
Forward/Backward Error

Suppose *want* to compute $y = f(x)$, but *approximate* $\hat{y} = \hat{f}(x)$.

What are the forward error and the backward error?

Forward error: $|\Delta y| = |f(x) - \hat{f}(x)|$ ← abs.

$\frac{|\Delta y|}{|y|}$ ← rel.



Forward/Backward Error: Example

1.41421... ← true
1.4 ← approx

Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$.
What's the (magnitude of) the forward error?

$\Delta y = y - \hat{y} = 0.0142\ldots$

rel. fwd. error

$0.01\ldots$

$\log_{10} 0.01 = -2$

0.600 1234567000

0.600 1234

Forward/Backward Error: Example

Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$.
What's the (magnitude of) the backward error?

$\sqrt{\hat{x}} = \hat{y}$

$\sqrt{\hat{x}} = 1.4$

$\hat{x} = 1.4 = 1.96$

$x - \hat{x} = 0.04$ ← abs bw error

→ $\frac{|x - \hat{x}|}{|x|} = 0.02$

Forward/Backward Error: Observations

What do you observe about the relative manitude of the relative errors?

rel.bw 0.02 rel.fw 0.01...

Sensitivity and Conditioning

What can we say about amplification of error?

$$|\text{rel. fw. error}| \leq \kappa \, |\text{rel. bw. error}|$$

smallest number that does this: condition number

$$\text{cond} = \kappa = \max_x \frac{|\Delta y|/|y|}{|\Delta x|/|x|}$$

$$\frac{|\Delta x|}{|x|} = 10^{-6}$$

$$k = 10,000 = 10^4$$

$$\frac{|\Delta y|}{|y|} \leq 10^{-2}$$

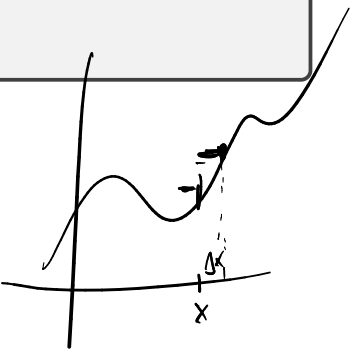
Example: Condition Number of Evaluating a Function

$$\Delta y = f(x+\Delta x) - f(x) \approx f'(x)\cdot \Delta x$$

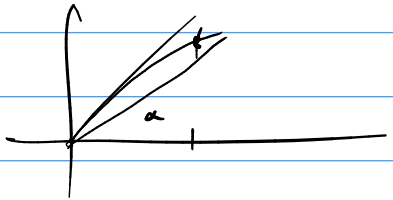
$y = f(x)$. Assume f differentiable.

Demo: Conditioning of Evaluating tan

$$\frac{|\Delta y|/|y|}{|\Delta x|/|x|} = \frac{|f'(x)|\,|\Delta x|/|f(x)|}{|\Delta x|/|x|} = \frac{|x|\,|f'(x)|}{|f(x)|}$$



$$\frac{|x|\,|\cos x|}{|\sin x|} \leq \frac{|x|}{\alpha|x|} \leq \frac{1}{\alpha}$$



Stability and Accuracy

Conditioning → problem itself
→ method

When is a method *accurate*?

Output is close to the true answer

When is a method *stable*?

"Backward stable"

A method is (bw-) stable if the result it produces is the exact solution to a nearby problem.

Getting into Trouble with Accuracy and Stability

$p(x) = 5$

How can I produce inaccurate results?

Apply an inaccurate method

Apply an unstable method to a well-conditioned problem

Apply any type of method to an ill-conditioned input

In-Class Activity: Forward/Backward Error

In-class activity: Forward/Backward Error

Wanted: Real Numbers... in a computer

Computers can represent *integers*, using bits:

$$23 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = (10111)_2$$

How would we represent fractions?

Fixed-Point Numbers

Suppose we use units of 64 bits, with 32 bits for exponents $\geqslant 0$ and 32 bits for exponents < 0. What numbers can we represent?

How many 'digits' of relative accuracy (think relative rounding error) are available for the smallest vs. the largest number?