Today

- cond. nr
- solving

Announcements

- Examlet 0
- HW2
- Python Workshop
- Piazza personal msgs
- κ(non-invertible)

If A is not invertible,
$\kappa(A) = \infty$

$\hookrightarrow \kappa(A) = \|A\| \, \|A^{-1}\|$

Conditioning of Linear Systems: More properties

- cond is relative to a given norm. So, to be precise, use

$$\text{cond}_2 \quad \text{or} \quad \text{cond}_\infty .$$

- If A^{-1} does not exist: $\text{cond}(A) = \infty$ by convention.

What is $\kappa(A^{-1})$?

$\kappa(A)$

What is the condition number of matrix-vector multiplication?

$\kappa(A)$

Demo: Condition number visualized

Demo: Conditioning of 2x2 Matrices

Residual Vector

What is the residual vector of solving the linear system

$$\mathbf{b} = A\mathbf{x}?$$

$r = b - Ax$

Residual and Error: Relationship

How do the (norms of the) residual vector $\mathbf{r}$ and the error $\Delta\mathbf{x} = \mathbf{x} - \widehat{\mathbf{x}}$ relate to one another?

$$\|\Delta x\| = \|x - \hat{x}\|$$
$$= \|A^{-1}(b - A\hat{x})\|$$
$$= \|A^{-1}r\|$$

Divide by $\|\hat{x}\|$:

$$\frac{\|\Delta x\|}{\|\hat{x}\|} = \frac{\|A^{-1}r\|}{\|\hat{x}\|} \le \frac{\|A^{-1}\|\|r\|}{\|\hat{x}\|} = \operatorname{cond}(A)\frac{\|r\|}{\|A\|\|\hat{x}\|}.$$

rel err $\qquad$ cond $\cdot$ rel resid

Changing the Matrix

So far, all our discussion was based on changing the right-hand side, i.e.

$$\cancel{A\mathbf{x} = \mathbf{b} \quad \to \quad A\widehat{\mathbf{x}} = \widehat{\mathbf{b}}.}$$

The matrix consists of FP numbers, too–it, too, is approximate. I.e.

$$A\mathbf{x} = \mathbf{b} \quad \to \quad \hat{A}\widehat{\mathbf{x}} = \mathbf{b}. \qquad \Delta A = \hat{A} - A$$

What can we say about the error now?

$$\Delta x = \hat{x} - x = A^{-1}(A\hat{x} - b) = A^{-1}(A\hat{x} - \underbrace{\hat{A}\hat{x}}_{(A - \Delta A)\hat{x}}) = -A^{-1}\Delta A\,\hat{x}$$

$$b = \hat{A}\hat{x}$$

$$\|\Delta x\| \le \|A^{-1}\|\,\|\Delta A\|\,\|\hat{x}\| \Rightarrow \frac{\|\Delta x\|}{\|\hat{x}\|} \le \operatorname{cond}(A)\frac{\|\Delta A\|}{\|A\|}$$

Changing Condition Numbers

Once we have a matrix A in a linear system $A\mathbf{x} = \mathbf{b}$, are we stuck with its condition number? Or could we improve it?

$DAx = Db$ ← row rescaling

$\underbrace{AD}y = b$ ← column rescaling

↪ $Dy = x$

What is this called as a general concept?

$MAx = Mb$ ← "left precond."

$AMy = b$

↪ $My = x$

In-Class Activity: Matrix Norms and Conditioning

In-class activity: Matrix Norms and Conditioning

Solving Systems: Triangular matrices

Solve

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

"back substitution"

Demo: Coding back-substitution

What about non-triangular matrices?

Gauss elim in a can: LU

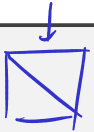
Gaussian Elimination

Demo: Vanilla Gaussian Elimination

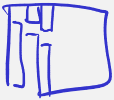
What do we get by doing Gaussian Elimination?

REF

How is that different from being upper triangular?



What if we do not just eliminate downward but also upward?



Gauss-Jordan elimination

do not use

Elimination Matrices

$M^{-1} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ +\frac{1}{2} & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$

What does this matrix do?

col 1 row 3

$$\underbrace{\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ -\frac{1}{2} & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}}_{A} \begin{array}{l} \\ -\frac{1}{2} \\ \leftarrow \\ \\ \\ \end{array} =$$

About Elimination Matrices

Are elimination matrices invertible?

✓

More on Elimination Matrices

Demo: Elimination matrices I

Idea: With enough elimination matrices, we should be able to get a matrix into row echelon form.

$$\underbrace{M_k \cdots M_2 M_1}_{\text{still lower } \Delta} A = U$$

So what do we get from many combined elimination matrices like that?

$$A = \underbrace{M_1^{-1} \cdots M_k^{-1}}_{\text{still } \Delta} U = LU$$

Demo: Elimination Matrices II

Summary on Elimination Matrices

- El.matrices with off-diagonal entries in a single column just "merge" when multiplied by one another.
- El.matrices with off-diagonal entries in different columns merge when we multiply (left-column) * (right-column) but not the other way around.
- Inverse: Flip sign below diagonal

LU Factorization

Can build a *factorization* from elimination matrices. How?

Solving $A\mathbf{x} = \mathbf{b}$

$A = LU$

Does LU help solve $A\mathbf{x} = \mathbf{b}$?

$n \times n$

$O(n^2)$

$+$ work to get LU.

$O(n^3)$

$L(\underbrace{Ux}_{y}) = b$

$Ly = b$ ← fw. subst.

$Ux = y$ ← bw. subst.

Demo: LU factorization

LU: Failure Cases?

$A \cdot B = C$

Is LU/Gaussian Elimination bulletproof?

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ & u_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$u_{11} \cdot 1 = 0 \rightarrow u_{11} = 0$

$\underbrace{l_{21} \cdot u_{11}}_{0} + \underbrace{1 \cdot 0}_{0} = 2$

Saving the LU Factorization

What can be done to get something *like* an LU factorization?

"partial" pivoting, by swapping rows

"complete" pivoting, by swapping rows & cols

not common

Recap: Permutation Matrices

How do we capture 'row switches' in a factorization?

$$\underbrace{\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}}_{P} \begin{pmatrix} A & A & A & A \\ B & B & B & B \\ C & C & C & C \\ D & D & D & D \end{pmatrix} = \begin{pmatrix} A & A & A & A \\ C & C & C & C \\ B & B & B & B \\ D & D & D & D \end{pmatrix}.$$

P is called a *permutation matrix*.

Q: What's P^{-1}? $= P$

Fixing nonexistence of LU

What does LU with permutations process look like?

Demo: LU with Partial Pivoting (Part I)

$$\cdots \underbrace{M_3 \, P_3 \, M_2 P_2 \, M_1 \, P_1} \, A$$

What about the L in LU?

Sort out what LU with pivoting looks like. Have: $M_3 P_3 M_2 P_2 M_1 P_1 A = U$.

Demo: LU with Partial Pivoting (Part II)