Today

- Householder
- Givens
- least squares w/ A not full rank
- SVD
- eigenvalues

Announcements:

- Examlet 1
- HW 4

Householder Transformations

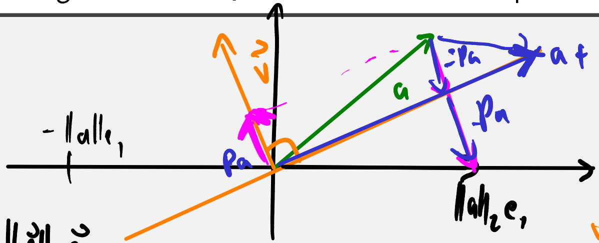
Reflection

$A = QR \rightsquigarrow Q^T A = R$

$(a-b)(a+b) = a^2 - b^2$

Find an *orthogonal* matrix Q to zero out the lower part of a vector a.

$[\, A \,]$ $\leftarrow \vec{a}$



$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\rightarrow Q^T \vec{a} = \pm \|\vec{a}\|_2 \vec{e}_1$

$\begin{pmatrix} \vdots \end{pmatrix} \quad \begin{pmatrix} \cdot \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$H = I - 2\frac{vv^T}{\underbrace{v^Tv}}$

Projection onto span v

$\rightarrow \overbrace{\underbrace{a - \|a\|_2 e_1}}^{v} \cdot (a + \|a\|_2 e_1) = 0$

$\|a\|_2^2 - \|a\|_2^2 = 0$

$\|v\|_2^2 = v^T v$

$Ha = \|a\|_2 e_1$

$O(n^2) \rightarrow (vv^T)x$

$O(n) \rightarrow v(v^T x)$

Householder Reflectors: Properties

Seen from picture (and easy to see with algebra):

$$H\mathbf{a} = \mathbf{e}_1.$$

$\|a\|_2$

Remarks:

- Q: What if we want to zero out only the $i+1$th through nth entry?
  A: Use $\mathbf{e}_i$ above.
- A product $H_n \cdots H_1 A = R$ of Householders makes it easy (and quite efficient!) to build a QR factorization.
- It turns out $\mathbf{v}' = \mathbf{a} + \|\mathbf{a}\|_2 \mathbf{e}_1$ works out, too–just pick whichever one causes less cancellation.
- H is symmetric
- H is orthogonal

Demo: 3x3 Householder demo

Givens Rotations

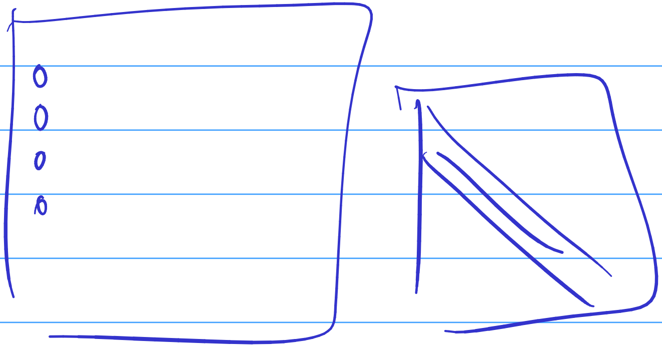
If reflections work, can we make rotations work, too?

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{pmatrix} \pm\sqrt{a_1^2 + a_2^2} \\ 0 \end{pmatrix}$$

$$c^2 + s^2 = 1$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} b \\ -a \end{pmatrix} = 0$$

Demo: 3x3 Givens demo

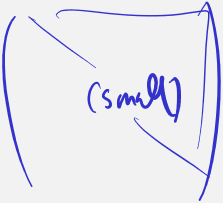


Rank-Deficient Matrices and QR

What happens with QR for rank-deficient matrices?

$$Ax \cong b$$

Q



$$A\vec{n} = \vec{0}$$

$$n \neq 0$$

$$\min \|Ax - b\|_2$$

also: $\min \|x\|_2$

total least squares

$\tilde{x}$ min. $\|Ax - b\|_2$

$\tilde{x} + n$

Rank-Deficient Matrices and Least-Squares

What happens with Least Squares for rank-deficient matrices?

$$A\mathbf{x} \cong \mathbf{b}$$

- QR still finds a solution with minimal residual
- By QR it's easy to see that least squares with a short-and-fat matrix is equivalent to a rank-deficient one.
- But: No longer unique. $\mathbf{x} + \mathbf{n}$ for $\mathbf{n} \in N(A)$ has the same residual.
- In other words: Have more freedom
  Or: Can demand another condition, for example:
  - Minimize $\|\mathbf{b} - A\mathbf{x}\|_2^2$, *and*
  - minimize $\|\mathbf{x}\|_2^2$, simultaneously.
    Unfortunately, QR does not help much with that $\rightarrow$ Need better tool.

Singular Value Decomposition (SVD)

What is the *Singular Value Decomposition* of an $m \times n$ matrix?

$$A = U \Sigma V^T$$

left, right sing vec → U, V orthogonal ← full

have orthonormal cols

econo

sing. values → Σ diag

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \qquad \sigma_1 \geq \sigma_2 \geq \sigma_n \geq 0$$

$$A^{-1} = V \frac{1}{\Sigma} U^T$$

SVD: What's this thing good for? (I)

$\|A\|_2 = \sigma_1$

$\mathrm{cond}(A) = \sigma_1 / \sigma_n$

$\mathrm{rank}(A) = \#$ nonzero singular

$\mathrm{num\,rank}(A, \epsilon) = \#$ singular values $> \epsilon$

SVD: What's this thing good for? (II)

- *Low-rank Approximation*

Eckart - Young - Mirsky

SVD: What's this thing good for? (III)

- The minimum norm solution to $A\mathbf{x} \cong \mathbf{b}$:

SVD: Minimum-Norm, Pseudoinverse

$\mathbf{y} = \Sigma^+ U^T \mathbf{b}$ is the minimum norm-solution to $\Sigma \mathbf{y} \cong U^T \mathbf{b}$.
Observe $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2$.

$$\mathbf{x} = V \Sigma^+ U^T \mathbf{b}$$

solves the minimum-norm least-squares problem.

Define $A^+ = V \Sigma^+ U^T$ and call it the pseudoinverse of A.
Coincides with prior definition in case of full rank.