Today:

- QR it.
- Krylov spaces
- SVD

- nonlinear

Announcements:

HW 6

Exam let 2

QR Iteration: Computational Expense

A full QR factorization at each iteration costs $O(n^3)$–can we make that cheaper?

same eigvals

$A = Q \quad Q^*$ ← Schur form, hard

→ $A = Q \quad Q^T$ ← upper Hessenberg

QR it applied to upper Hessenberg: "$O(n^3)$" ← assuming n iteration

GivenQR applied to upper Hessenberg: $O(n^2)$

Demo: Householder Similarity Transforms

QR/Hessenberg: Overall procedure

Overall procedure:

1. Reduce matrix to Hessenberg form (using Householder)
2. Apply QR iteration using Givens QR to obtain Schur form

For symmetric matrices:

- Use Householders to attain tridiagonal form
- Use QR iteration with Givens to attain diagonal form

Krylov space methods: Intro

What subspaces can we use to look for eigenvectors?

starting vec $\tilde{x}$

$$\operatorname{span}(\underbrace{\tilde{x}}_{x_0}, \underbrace{A\tilde{x}}_{x_1}, \underbrace{A^2\tilde{x}}_{x_2}, A^3\tilde{x}, \dots) \leftarrow \text{Krylov space}$$

$$K_k = \begin{pmatrix} | & & | \\ x_0 & \cdots & x_{k-1} \\ | & & | \end{pmatrix} \quad (n \times k)$$

$$x_0 \quad Ax_0 \quad \cdots \quad A^{n-1}x_0$$

Krylov for Matrix Factorization

What matrix factorization is obtained through Krylov space methods?

$$A K_n = K_n \underbrace{\begin{pmatrix} 0 & 0 & & 0 & | \\ 1 & 0 & & | & \\ 0 & 1 & & | & K_n^{-1}x_n \\ \vdots & \vdots & \ddots & 0 & \\ 0 & 0 & & 1 & | \end{pmatrix}}_{\text{upper Hessenberg} \to C_n} \Bigg| K_n^{-1}.$$

$$\underbrace{K_n^{-1} A K_n}_{} = C_n$$

similarity t $\to$ upper Hess.

Conditioning in Krylov Space Methods/Arnoldi Iteration (I)

What is a problem with Krylov space methods? How can we fix it?

$$Q_n R_n = K_n \longrightarrow Q_n = K_n R_n^{-1}$$

$$Q_n^T = Q_n^{-1} = R_n K_n^{-1}$$

$$\left(\,|||\,\right)\left(\triangledown\right) = \left(K_n\right)$$

$$Q_n^T A Q_n = \underbrace{\underset{\triangledown}{R_n}\;\underbrace{K_n^{-1} A K_n}_{C_n}\;\underset{\triangledown}{R_n^{-1}}}_{\text{upper Hessenberg}}$$

$$H$$

$$K_n^{-1} A K_n = C_n$$

Conditioning in Krylov Space Methods/Arnoldi Iteration (II)

$$Aq_1 = h_{11} q_1 + h_{21} q_2$$

$$Aq_k = \underbrace{h_{1k} \cdot q_1 + \cdots + h_{k+1,k} q_k}$$

$$h_{jk} = q_j^T (Aq_k)$$

$$Q_n = (q_1 \cdots q_n)$$

$$Q_n^T A Q_n = H \quad / \, Q_n \cdot$$

$$\Rightarrow A Q_n = Q_n H$$

$$A Q_n = Q_n$$

$$(H)_{ij} = h_{ij}$$

Demo: Arnoldi Iteration (Part 1)

ARPACK

Krylov: What about eigenvalues?

How can we use Arnoldi/Lanczos to compute eigenvalues?

Demo: Arnoldi Iteration (Part 2)

Computing the SVD (Kiddy Version)

How can I compute an SVD of a matrix A?

Demo: Computing the SVD

In-Class Activity: Eigenvalue Computations

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