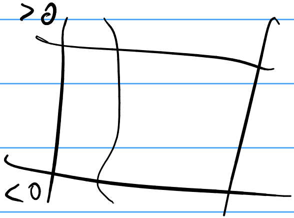
Today

- Fixed point
- Newton
- Quasi-Newton
- nD → Newton

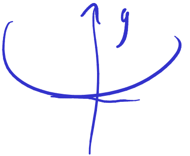
Announcements

- Exam left
- 4CH
- Break



Fixed Point Iteration

$e_{k+1} \le C \cdot e_k^r$



$g'(x^*) = 0$

$\Rightarrow$ good (because quad)

$$
\begin{aligned}
x_0 &= \langle \text{starting guess} \rangle \\
x_{k+1} &= g(x_k)
\end{aligned}
$$

Demo: Fixed point iteration

When does fixed point iteration converge? Assume g is smooth.

$x^* = fp$

$e_k = x_k - x^*$

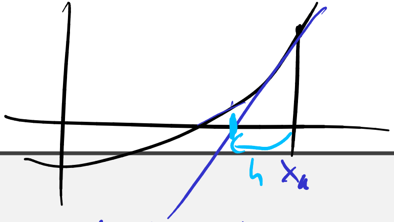
$\rightarrow e_{k+1} = g'(\theta_k) \cdot e_k$

somewhere between $[x^*, x_k]$

(at least) linear if $|g'| < 1$ near x^*

(at least) quadratic if $g'(x^*) = 0$

Newton's Method



Derive Newton's method.

$$0 = f(x_k+h) \approx f(x_k) + \underbrace{f'(x_k)\cdot h} + \cancel{HOT}$$

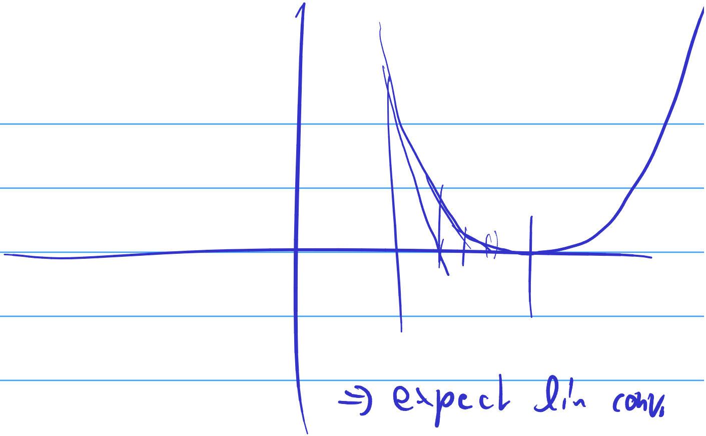
$$0 = f(x_k) + f'(x_k)\cdot h$$

$$-f(x_k) = f'(x_k)\cdot h$$

$$-\frac{f(x_k)}{f'(x_k)} = h$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = g(x_k)$$

- has a FP at root
- multiple roots $\Rightarrow$ trouble



⇒ expect lin conv.

Convergence and Properties of Newton

What's the rate of convergence of Newton's method?

$g'(x^*) = 0$?

$\Rightarrow$ yep

$g'(x) = \dfrac{f(x)\, f''(x)}{f'(x)^2}$ $\Rightarrow$ quad. conv.

Drawbacks of Newton?

- need derivative
- locally convergent

Demo: Newton's method

Demo: Convergence of Newton's Method

Deflation

Found root ξ_1 of f

Would like f_1 with all the roots of f except for ξ_1.

$$f(x) = (x-\xi_1)(x-\xi_2)(x-\xi_3)$$

$$f_1(x) = \frac{f(x)}{x-\xi_1}$$

[illegible]

Secant Method

guesses: x_0, x_1

What would Newton without the use of the derivative look like?

$$s = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$x_2 = x_1 - \frac{f(x_1)}{s}$$

Convergence of Properties of Secant

Rate of convergence (not shown) is $\left(1+\sqrt{5}\right)/2 \approx 1.618$.

Drawbacks of Secant?

- still locally conv.
- two starting guesses
- slower than Newton

Demo: Secant Method

Demo: Convergence of the Secant Method

Secant (and similar methods) are called Quasi-Newton Methods.

Root Finding with Interpolants

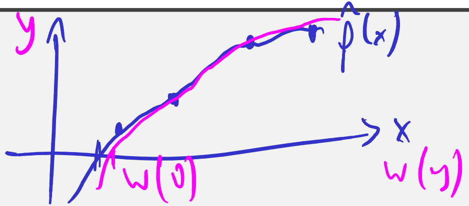
Secant method uses a linear interpolant based on points $f(x_k)$, $f(x_{k-1})$, could use more points and higher-order interpolant:

use not line through $x_k \ldots x_0$ but parabola

→ "Muller's method"

conv. rate: 1.81

What about existence of roots in that case?



"Inverse quadratic interpolation"

Achieving Global Convergence

The linear approximations in Newton and Secant are only good locally. How could we use that?

- stop Newton from going crazy
- limiting step size
- trust region
- hybrid methods

Fixed Point Iteration

$$\begin{aligned} \mathbf{x}_0 &= \langle \text{starting guess} \rangle \\ \mathbf{x}_{k+1} &= \mathbf{g}(\mathbf{x}_k) \end{aligned}$$

When does this converge?

Newton's Method

What does Newton's method look like in n dimensions?

Downsides of n-dim. Newton?

Demo: Newton's method in n dimensions