Today:

- IVP
  - ↪ Zoology
  - ↪ methods

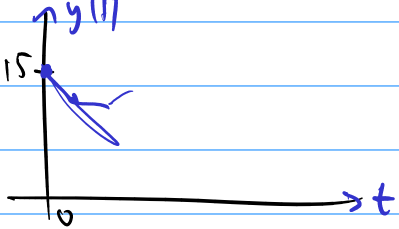
Announcements:

- HW 12 (today, lighter)
- Exam let 4

$$y'(t) = f(y(t))$$

$$y(0) = 15$$

$$y(t) = y(0) + \int_0^t f(y(t)) = y(0) + \sum \omega_i f(y(t_i))$$



$$y(t) = y(0) + \int_0^t f(y(\tau))\, d\tau$$

$$= y(0) + \underbrace{(t-0)}_{\Delta t} f(0)$$

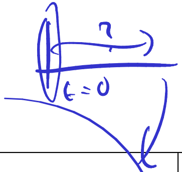
What can we solve already?

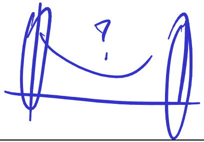
- Linear Systems: yes
- Nonlinear systems: yes
- Systems with derivatives: no

ordinary

partial ("PDEs")

Some Applications





IVPs	BVPs
▶ Population dynamics $y_1' = y_1(\alpha_1 - \beta_1 y_2)$ (prey) $y_2' = y_2(-\alpha_2 + \beta_2 y_1)$ (predator) ▶ chemical reactions ▶ equations of motion	▶ bridge load ▶ pollutant concentration (steady state) ▶ temperature (steady state)

Initial Value Problems: Problem Statement

Want: Function $\mathbf{y} : [0, T] \to \mathbb{R}^n$ so that

- $\mathbf{y}^{(k)}(t) = \mathbf{f}(t, \mathbf{y}, \mathbf{y}', \mathbf{y}'', \dots, \mathbf{y}^{(k-1)})$ (*explicit*)

  or
- $\mathbf{f}(t, \mathbf{y}, \mathbf{y}', \mathbf{y}'', \dots, \mathbf{y}^{(k)}) = \mathbf{0}$ (*implicit*)

are called explicit/implicit *kth-order ordinary differential equations* (*ODEs*).
Give a simple example.

$y' = \alpha y$ $\quad$ $y(t) = \exp(\alpha t)$

Not uniquely solvable on its own. What else is needed?

$\vec{y}(0), \vec{y}^{(1)}(0), \dots, \vec{y}^{(k-1)}$

Reducing ODEs to First-Order Form

$\vec{w}'(t) = \tilde{f}(\vec{w})$

A kth order ODE can always be reduced to first order. Do this in this example:

$$y''(t) = f(y(t))$$

$\vec{w} = \begin{pmatrix} u \\ v \end{pmatrix}$

$u = y$

$v = y' = u'$

$$\vec{u}' = \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ f(u) \end{pmatrix} = \tilde{f}(\vec{w})$$

↳ By introducing extra variables for derivatives, we can always rewrite to first-order form.

Properties of ODEs

$\vec{y}'(t) = f(t, \vec{y}(t))$

What is a linear ODE?

$f(\vec{y}, t) = A(t)\,\vec{y} + \vec{b}$

What is a linear and homogeneous ODE?

$f(\vec{y}, t) = A(t)\,\vec{y}$

What is a constant-coefficient ODE?

$f(\vec{y}, t) = A\vec{y}$

Properties of ODEs (II)

general class of ODEs → $y'(t) = f(y)$

"formula for f has no t in it"

What is an autonomous ODE?

$$\boxed{f(\vec{y}, t) = \bar{f}(\vec{y})}$$

make an auxiliary variable $\tilde{t} = y_{n+1}$

$y_1, \ldots, y_n$

$y_{n+1}' = 1$

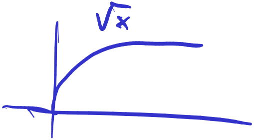
$y_{n+1}(t_0) = t_0$

WLOG

$f(\vec{y})$

Existence and Uniqueness

Consider the perturbed problem



$$\begin{cases} \mathbf{y}'(t) = \mathbf{f}(\mathbf{y}) \\ \mathbf{y}(t_0) = \mathbf{y}_0 \end{cases} \qquad \begin{cases} \widehat{\mathbf{y}}'(t) = \mathbf{f}(\widehat{\mathbf{y}}) \\ \widehat{\mathbf{y}}(t_0) = \widehat{\mathbf{y}}_0 \end{cases}$$

Then if $\mathbf{f}$ is *Lipschitz continuous* (has 'bounded slope'), i.e.

$$\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\widehat{\mathbf{y}})\| \leqslant L \|\mathbf{y} - \widehat{\mathbf{y}}\|$$

(where L is called the *Lipschitz constant*), then...

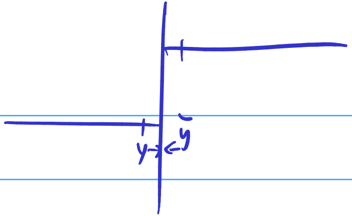
with t

there exists a solution y(t) in a neighborhood of t_0

$$\|y(t) - \hat{y}(t)\| \le e^{L(t-t_0)} \|\vec{y}_0 - \hat{\vec{y}}_0\|$$

What does this mean for uniqueness?

$\Rightarrow$ implies uniqueness because $\|\vec{y}_0 - \hat{\vec{y}}_0\| = 0$



$$\|f(y) - f(\bar{y})\| \le C\,|y - \bar{y}|$$

$f(y) = y^2$ $\qquad$ $f(y) = y$

Let $y \in [-10{,}000,\ 10{,}000]$

Conditioning

Unfortunate terminology accident: "Stability" in ODE-speak
To adapt to conventional terminology, we will use 'Stability' for

- the conditioning of the IVP, *and*
- the stability of the methods we cook up.

Some terminology:

An ODE is stable if and only if...

For all $\varepsilon > 0$ there exists a $\delta > 0$ so that

$$\|\hat{y}_0 - y_0\| < \delta \Rightarrow \|y(t) - \hat{y}(t)\|_\infty < \varepsilon$$

for all $t \geq t_0$

An ODE is asymptotically stable if and only if

$$\|\hat{y}(t) - y(t)\| \to 0 \quad (t \to \infty)$$

$f : \mathbb{R} \to \mathbb{R}$ continuous

$\iff \forall \varepsilon > 0 \ \exists \delta > 0 :$

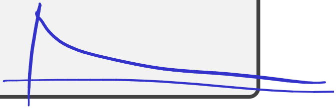
$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Example I: Scalar, Constant-Coefficient

$$\begin{cases} y'(t) = \lambda y \\ y(0) = y_0 \end{cases} \qquad \text{where} \quad \lambda = a + ib$$

Solution?

$y(t) = e^{\lambda t} \cdot y_0$



When is this stable?

For $a > 0$: unstable

For $a \le 0$ stable

Example II: Constant-Coefficient System

$$\begin{cases} \mathbf{y}'(t) = A\mathbf{y}(t) \\ \mathbf{y}(t_0) = \mathbf{y}_0 \end{cases}$$

$w_i' = \lambda_i w_i$

stable iff $\mathrm{Re}\,\lambda_i \le 0$

Assume $V^{-1}AV = D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ diagonal.

How do we find a solution?

$Vw = y$

$$w(t) := V^{-1} y(t)$$

$$w'(t) = V^{-1} y'(t) = V^{-1} A y = \underbrace{V^{-1} A V}_{D} w$$

When is this stable?

stable unstable

$$\mathrm{Re}\,\lambda_i \le 0$$

Euler's Method

Forward Euler

Discretize the IVP

$$\begin{cases} \mathbf{y}'(t) = \mathbf{f}(\mathbf{y}) \\ \mathbf{y}(t_0) = \mathbf{y}_0 \end{cases}$$

- Discrete times: $t_1, t_2, \ldots,$ with $t_{i+1} = t_i + h$
- Discrete function values: $\mathbf{y}_k \approx \mathbf{y}(t_k)$.

$$\vec{y}_{k+1} = \vec{y}_k + h f(\vec{y}_k)$$

Euler's method: Forward and Backward

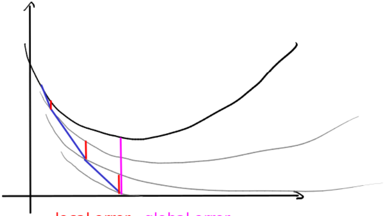
$$\mathbf{y}(t) = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{y}(\tau))\mathrm{d}\tau,$$

Use 'left rectangle rule' on integral:

Use 'right rectangle rule' on integral:

Demo: Forward Euler stability

Global and Local Error



Let $u_k(t)$ be the function that solves the ODE with the initial condition $u_k(t_k) = y_k$.

Define the local error at step k as...

Define the global error at step k as...