Scientific Computing: An Introductory Survey

Chapter 7 – Interpolation

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Outline

1. Interpolation
2. Polynomial Interpolation
3. Piecewise Polynomial Interpolation

Interpolation

- Basic interpolation problem: for given data

  $$(t_1, y_1),\ (t_2, y_2),\ \ldots\ (t_m, y_m) \quad \text{with} \quad t_1 < t_2 < \cdots < t_m$$

  determine function $f\colon \mathbb{R} \to \mathbb{R}$ such that

  $$f(t_i) = y_i, \quad i = 1, \ldots, m$$

- f is *interpolating function*, or *interpolant*, for given data
- Additional data might be prescribed, such as slope of interpolant at given points
- Additional constraints might be imposed, such as smoothness, monotonicity, or convexity of interpolant
- f could be function of more than one variable, but we will consider only one-dimensional case

Purposes for Interpolation

- Plotting smooth curve through discrete data points
- Reading between lines of table
- Differentiating or integrating tabular data
- Quick and easy evaluation of mathematical function
- Replacing complicated function by simple one

Interpolation vs Approximation

- By definition, interpolating function fits given data points exactly
- Interpolation is inappropriate if data points subject to significant errors
- It is usually preferable to smooth noisy data, for example by least squares approximation
- Approximation is also more appropriate for special function libraries

Issues in Interpolation

Arbitrarily many functions interpolate given set of data points

- What form should interpolating function have?
- How should interpolant behave between data points?
- Should interpolant inherit properties of data, such as monotonicity, convexity, or periodicity?
- Are parameters that define interpolating function meaningful?
- If function and data are plotted, should results be visually pleasing?

Choosing Interpolant

Choice of function for interpolation based on

- How easy interpolating function is to work with
    - determining its parameters
    - evaluating interpolant
    - differentiating or integrating interpolant
- How well properties of interpolant match properties of data to be fit (smoothness, monotonicity, convexity, periodicity, etc.)

Functions for Interpolation

- Families of functions commonly used for interpolation include
  - Polynomials
  - Piecewise polynomials
  - Trigonometric functions
  - Exponential functions
  - Rational functions
- For now we will focus on interpolation by polynomials and piecewise polynomials
- We will consider trigonometric interpolation (DFT) later

Basis Functions

- Family of functions for interpolating given data points is spanned by set of *basis functions* $\phi_1(t), \dots, \phi_n(t)$
- Interpolating function f is chosen as linear combination of basis functions,

$$f(t) = \sum_{j=1}^{n} x_j \phi_j(t)$$

- Requiring f to interpolate data (t_i, y_i) means

$$f(t_i) = \sum_{j=1}^{n} x_j \phi_j(t_i) = y_i, \quad i = 1, \dots, m$$

which is system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ for n-vector $\boldsymbol{x}$ of parameters x_j, where entries of $m \times n$ matrix $\boldsymbol{A}$ are given by $a_{ij} = \phi_j(t_i)$

Existence, Uniqueness, and Conditioning

- Existence and uniqueness of interpolant depend on number of data points m and number of basis functions n
- If $m > n$, interpolant usually doesn't exist
- If $m < n$, interpolant is not unique
- If $m = n$, then basis matrix $\boldsymbol{A}$ is nonsingular provided data points t_i are distinct, so data can be fit exactly
- Sensitivity of parameters $\boldsymbol{x}$ to perturbations in data depends on $\mathrm{cond}(\boldsymbol{A})$, which depends in turn on choice of basis functions

Polynomial Interpolation

- Simplest and most common type of interpolation uses polynomials
- Unique polynomial of degree at most $n-1$ passes through n data points (t_i, y_i), $i = 1, \ldots, n$, where t_i are distinct
- There are many ways to represent or compute interpolating polynomial, but in theory all must give same result

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Monomial Basis

- *Monomial basis functions*

  $$\phi_j(t) = t^{j-1}, \quad j = 1, \ldots, n$$

  give interpolating polynomial of form

  $$p_{n-1}(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$$

  with coefficients $\boldsymbol{x}$ given by $n \times n$ linear system

  $$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & t_1 & \cdots & t_1^{n-1} \\ 1 & t_2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & \cdots & t_n^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \boldsymbol{y}$$

- Matrix of this form is called *Vandermonde matrix*

Example: Monomial Basis

- Determine polynomial of degree two interpolating three data points $(-2,-27)$, $(0,-1)$, $(1,0)$
- Using monomial basis, linear system is

$$\boldsymbol{Ax} = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boldsymbol{y}$$

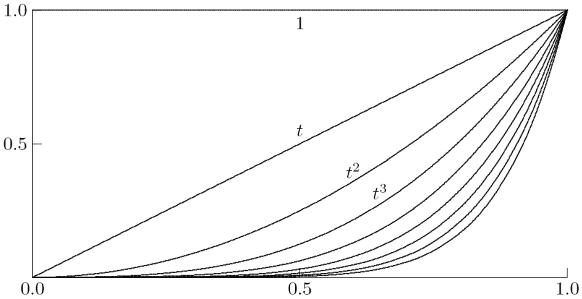
- For these particular data, system is

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -27 \\ -1 \\ 0 \end{bmatrix}$$

whose solution is $x = \begin{bmatrix} -1 & 5 & -4 \end{bmatrix}^T$, so interpolating polynomial is

$$p_2(t) = -1 + 5t - 4t^2$$

Monomial Basis, continued



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- Solving system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ using standard linear equation solver to determine coefficients $\boldsymbol{x}$ of interpolating polynomial requires $\mathcal{O}(n^3)$ work

Monomial Basis, continued

- For monomial basis, matrix $\boldsymbol{A}$ is increasingly ill-conditioned as degree increases
- Ill-conditioning does not prevent fitting data points well, since residual for linear system solution will be small
- But it does mean that values of coefficients are poorly determined
- Both conditioning of linear system and amount of computational work required to solve it can be improved by using different basis
- Change of basis still gives same interpolating polynomial for given data, but *representation* of polynomial will be different

Monomial Basis, continued

- Conditioning with monomial basis can be improved by shifting and scaling independent variable t

  $$\phi_j(t) = \left(\frac{t-c}{d}\right)^{j-1}$$

  where, $c = (t_1 + t_n)/2$ is midpoint and $d = (t_n - t_1)/2$ is half of range of data
- New independent variable lies in interval $[-1, 1]$, which also helps avoid overflow or harmful underflow
- Even with optimal shifting and scaling, monomial basis usually is still poorly conditioned, and we must seek better alternatives

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Evaluating Polynomials

- When represented in monomial basis, polynomial

  $$p_{n-1}(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$$

  can be evaluated efficiently using *Horner's nested evaluation* scheme

  $$p_{n-1}(t) = x_1 + t(x_2 + t(x_3 + t(\cdots(x_{n-1} + t x_n)\cdots)))$$

  which requires only n additions and n multiplications
- For example,

  $$1 - 4t + 5t^2 - 2t^3 + 3t^4 = 1 + t(-4 + t(5 + t(-2 + 3t)))$$

- Other manipulations of interpolating polynomial, such as differentiation or integration, are also relatively easy with monomial basis representation

Lagrange Interpolation

- For given set of data points (t_i, y_i), $i = 1, \ldots, n$, *Lagrange basis functions* are defined by

$$\ell_j(t) = \prod_{k=1, k \neq j}^{n} (t - t_k) \,/\, \prod_{k=1, k \neq j}^{n} (t_j - t_k), \quad j = 1, \ldots, n$$

- For Lagrange basis,

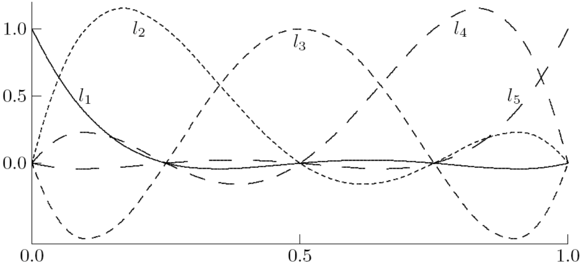
$$\ell_j(t_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad i, j = 1, \ldots, n$$

  so matrix of linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ is identity matrix

- Thus, Lagrange polynomial interpolating data points (t_i, y_i) is given by

$$p_{n-1}(t) = y_1 \ell_1(t) + y_2 \ell_2(t) + \cdots + y_n \ell_n(t)$$

Lagrange Basis Functions



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- Lagrange interpolant is easy to determine but more expensive to evaluate for given argument, compared with monomial basis representation
- Lagrangian form is also more difficult to differentiate, integrate, etc.

Example: Lagrange Interpolation

- Use Lagrange interpolation to determine interpolating polynomial for three data points $(-2,-27)$, $(0,-1)$, $(1,0)$
- Lagrange polynomial of degree two interpolating three points (t_1, y_1), (t_2, y_2), (t_3, y_3) is given by $p_2(t) =$

$$y_1 \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)} + y_2 \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)} + y_3 \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}$$

- For these particular data, this becomes

$$p_2(t) = -27 \frac{t(t-1)}{(-2)(-2-1)} + (-1) \frac{(t+2)(t-1)}{(2)(-1)}$$

Newton Interpolation

- For given set of data points (t_i, y_i), $i = 1, \ldots, n$, *Newton basis functions* are defined by

  $$\pi_j(t) = \prod_{k=1}^{j-1} (t - t_k), \quad j = 1, \ldots, n$$

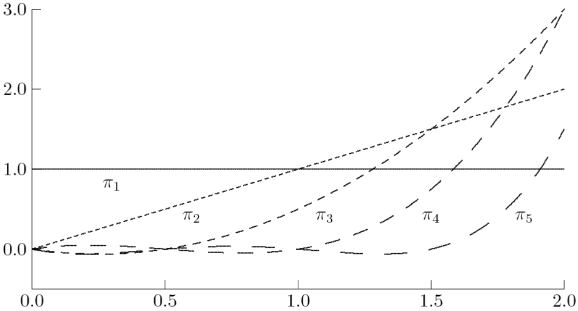
  where value of product is taken to be 1 when limits make it vacuous

- Newton interpolating polynomial has form

  $$\begin{aligned} p_{n-1}(t) \quad &= \quad x_1 + x_2(t - t_1) + x_3(t - t_1)(t - t_2) + \\ &\quad \cdots + x_n(t - t_1)(t - t_2) \cdots (t - t_{n-1}) \end{aligned}$$

- For $i < j$, $\pi_j(t_i) = 0$, so basis matrix $\boldsymbol{A}$ is lower triangular, where $a_{ij} = \pi_j(t_i)$

Newton Basis Functions



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Newton Interpolation, continued

- Solution $\boldsymbol{x}$ to system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ can be computed by forward-substitution in $\mathcal{O}(n^2)$ arithmetic operations
- Moreover, resulting interpolant can be evaluated efficiently for any argument by nested evaluation scheme similar to Horner's method
- Newton interpolation has better balance between cost of computing interpolant and cost of evaluating it

Example: Newton Interpolation

- Use Newton interpolation to determine interpolating polynomial for three data points $(-2,-27)$, $(0,-1)$, $(1,0)$
- Using Newton basis, linear system is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & t_2 - t_1 & 0 \\ 1 & t_3 - t_1 & (t_3 - t_1)(t_3 - t_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

- For these particular data, system is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -27 \\ -1 \\ 0 \end{bmatrix}$$

whose solution by forward substitution is $x = \begin{bmatrix} -27 & 13 & -4 \end{bmatrix}^T$, so interpolating polynomial is

$$p(t) = -27 + 13(t+2) - 4(t+2)t$$

Newton Interpolation, continued

- If $p_j(t)$ is polynomial of degree $j-1$ interpolating j given points, then for any constant x_{j+1},

  $$p_{j+1}(t) = p_j(t) + x_{j+1}\pi_{j+1}(t)$$

  is polynomial of degree j that also interpolates same j points
- Free parameter x_{j+1} can then be chosen so that $p_{j+1}(t)$ interpolates y_{j+1},

  $$x_{j+1} = \frac{y_{j+1} - p_j(t_{j+1})}{\pi_{j+1}(t_{j+1})}$$

- Newton interpolation begins with constant polynomial $p_1(t) = y_1$ interpolating first data point and then successively incorporates each remaining data point into interpolant < interactive example >

Divided Differences

- Given data points (t_i, y_i), $i = 1, \ldots, n$, *divided differences*, denoted by $f[\,]$, are defined recursively by

  $$f[t_1, t_2, \ldots, t_k] = \frac{f[t_2, t_3, \ldots, t_k] - f[t_1, t_2, \ldots, t_{k-1}]}{t_k - t_1}$$

  where recursion begins with $f[t_k] = y_k$, $k = 1, \ldots, n$

- Coefficient of jth basis function in Newton interpolant is given by

  $$x_j = f[t_1, t_2, \ldots, t_j]$$

- Recursion requires $\mathcal{O}(n^2)$ arithmetic operations to compute coefficients of Newton interpolant, but is less prone to overflow or underflow than direct formation of triangular Newton basis matrix

Orthogonal Polynomials

- Inner product can be defined on space of polynomials on interval $[a, b]$ by taking

$$\langle p, q \rangle = \int_a^b p(t)q(t)w(t)dt$$

  where $w(t)$ is nonnegative *weight function*
- Two polynomials p and q are *orthogonal* if $\langle p, q \rangle = 0$
- Set of polynomials $\{p_i\}$ is *orthonormal* if

$$\langle p_i, p_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

- Given set of polynomials, Gram-Schmidt orthogonalization can be used to generate orthonormal set spanning same space

Orthogonal Polynomials, continued

- For example, with inner product given by weight function $w(t) \equiv 1$ on interval $[-1, 1]$, applying Gram-Schmidt process to set of monomials $1, t, t^2, t^3, \ldots$ yields *Legendre polynomials*

  $$1, \;\; t, \;\; (3t^2 - 1)/2, \;\; (5t^3 - 3t)/2, \;\; (35t^4 - 30t^2 + 3)/8,$$
  $$(63t^5 - 70t^3 + 15t)/8, \;\; \ldots$$

  first n of which form an orthogonal basis for space of polynomials of degree at most $n - 1$

- Other choices of weight functions and intervals yield other orthogonal polynomials, such as Chebyshev, Jacobi, Laguerre, and Hermite

Orthogonal Polynomials, continued

- Orthogonal polynomials have many useful properties
- They satisfy three-term recurrence relation of form

$$p_{k+1}(t) = (\alpha_k t + \beta_k) p_k(t) - \gamma_k p_{k-1}(t)$$

  which makes them very efficient to generate and evaluate
- Orthogonality makes them very natural for least squares approximation, and they are also useful for generating Gaussian quadrature rules, which we will see later

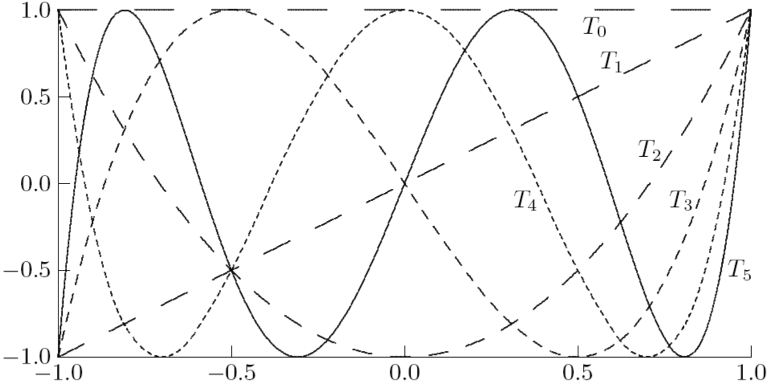
Chebyshev Polynomials

- kth *Chebyshev polynomial* of first kind, defined on interval $[-1, 1]$ by
$$T_k(t) = \cos(k \arccos(t))$$
are orthogonal with respect to weight function $(1 - t^2)^{-1/2}$

- First few Chebyshev polynomials are given by
$$1, \;\; t, \;\; 2t^2 - 1, \;\; 4t^3 - 3t, \;\; 8t^4 - 8t^2 + 1, \;\; 16t^5 - 20t^3 + 5t, \;\; \dots$$

- *Equi-oscillation property* : successive extrema of T_k are equal in magnitude and alternate in sign, which distributes error uniformly when approximating arbitrary continuous function

Chebyshev Basis Functions



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Chebyshev Points

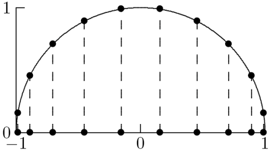
- *Chebyshev points* are zeros of T_k, given by

$$t_i = \cos\left(\frac{(2i-1)\pi}{2k}\right), \quad i = 1, \dots, k$$

  or extrema of T_k, given by

$$t_i = \cos\left(\frac{i\pi}{k}\right), \quad i = 0, 1, \dots, k$$

- Chebyshev points are abscissas of points equally spaced around unit circle in $\mathbb{R}^2$



- Chebyshev points have attractive properties for interpolation and other problems

Interpolating Continuous Functions

- If data points are discrete sample of continuous function, how well does interpolant approximate that function between sample points?

- If f is smooth function, and p_{n-1} is polynomial of degree at most $n-1$ interpolating f at n points $t_1, \ldots, t_n$, then

  $$f(t) - p_{n-1}(t) = \frac{f^{(n)}(\theta)}{n!}(t-t_1)(t-t_2)\cdots(t-t_n)$$

  where θ is some (unknown) point in interval $[t_1, t_n]$

- Since point θ is unknown, this result is not particularly useful unless bound on appropriate derivative of f is known

Interpolating Continuous Functions, continued

- If $|f^{(n)}(t)| \le M$ for all $t \in [t_1, t_n]$, and $h = \max\{t_{i+1} - t_i : i = 1, \ldots, n-1\}$, then

$$\max_{t \in [t_1, t_n]} |f(t) - p_{n-1}(t)| \le \frac{M h^n}{4n}$$

- Error diminishes with increasing n and decreasing h, but only if $|f^{(n)}(t)|$ does not grow too rapidly with n

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High-Degree Polynomial Interpolation

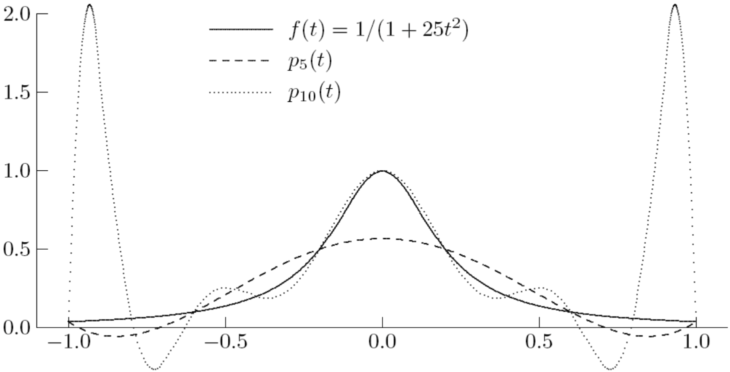
- Interpolating polynomials of high degree are expensive to determine and evaluate
- In some bases, coefficients of polynomial may be poorly determined due to ill-conditioning of linear system to be solved
- High-degree polynomial necessarily has lots of “wiggles,” which may bear no relation to data to be fit
- Polynomial passes through required data points, but it may oscillate wildly between data points

Convergence

- Polynomial interpolating continuous function may not converge to function as number of data points and polynomial degree increases
- Equally spaced interpolation points often yield unsatisfactory results near ends of interval
- If points are bunched near ends of interval, more satisfactory results are likely to be obtained with polynomial interpolation
- Use of Chebyshev points distributes error evenly and yields convergence throughout interval for any sufficiently smooth function

Example: Runge's Function

- Polynomial interpolants of Runge's function at *equally spaced* points *do not* converge



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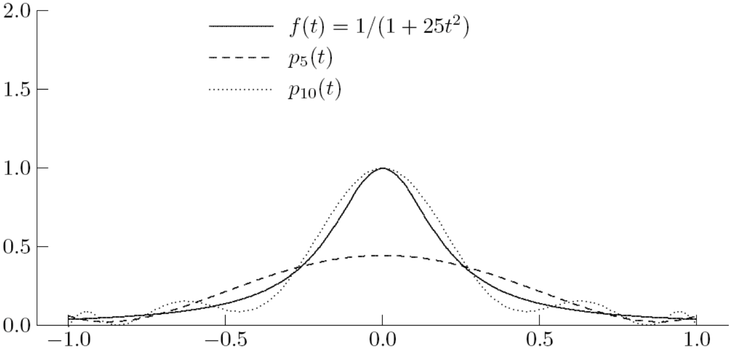
Example: Runge's Function

- Polynomial interpolants of Runge's function at *Chebyshev* points *do* converge



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Taylor Polynomial

- Another useful form of polynomial interpolation for smooth function f is polynomial given by truncated Taylor series

$$p_n(t) = f(a) + f'(a)(t-a) + \frac{f''(a)}{2}(t-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(t-a)^n$$

- Polynomial interpolates f in that values of p_n and its first n derivatives match those of f and its first n derivatives evaluated at $t = a$, so $p_n(t)$ is good approximation to $f(t)$ for t near a

- We have already seen examples in Newton's method for nonlinear equations and optimization

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Piecewise Polynomial Interpolation

- Fitting single polynomial to large number of data points is likely to yield unsatisfactory oscillating behavior in interpolant
- Piecewise polynomials provide alternative to practical and theoretical difficulties with high-degree polynomial interpolation
- Main advantage of piecewise polynomial interpolation is that large number of data points can be fit with low-degree polynomials
- In piecewise interpolation of given data points (t_i, y_i), *different* function is used in each subinterval $[t_i, t_{i+1}]$
- Abscissas t_i are called *knots* or *breakpoints*, at which interpolant changes from one function to another

Piecewise Interpolation, continued

- Simplest example is piecewise linear interpolation, in which successive pairs of data points are connected by straight lines
- Although piecewise interpolation eliminates excessive oscillation and nonconvergence, it appears to sacrifice smoothness of interpolating function
- We have many degrees of freedom in choosing piecewise polynomial interpolant, however, which can be exploited to obtain smooth interpolating function despite its piecewise nature

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Hermite Interpolation

- In *Hermite interpolation*, derivatives as well as values of interpolating function are taken into account
- Including derivative values adds more equations to linear system that determines parameters of interpolating function
- To have unique solution, number of equations must equal number of parameters to be determined
- Piecewise cubic polynomials are typical choice for Hermite interpolation, providing flexibility, simplicity, and efficiency

Hermite Cubic Interpolation

- *Hermite cubic interpolant* is piecewise cubic polynomial interpolant with continuous first derivative
- Piecewise cubic polynomial with n knots has $4(n-1)$ parameters to be determined
- Requiring that it interpolate given data gives $2(n-1)$ equations
- Requiring that it have one continuous derivative gives $n-2$ additional equations, or total of $3n-4$, which still leaves n free parameters
- Thus, Hermite cubic interpolant is not unique, and remaining free parameters can be chosen so that result satisfies additional constraints

Cubic Spline Interpolation

- *Spline* is piecewise polynomial of degree k that is $k-1$ times continuously differentiable
- For example, linear spline is of degree 1 and has 0 continuous derivatives, i.e., it is continuous, but not smooth, and could be described as “broken line”
- *Cubic spline* is piecewise cubic polynomial that is twice continuously differentiable
- As with Hermite cubic, interpolating given data and requiring one continuous derivative imposes $3n-4$ constraints on cubic spline
- Requiring continuous second derivative imposes $n-2$ additional constraints, leaving 2 remaining free parameters

Cubic Splines, continued

Final two parameters can be fixed in various ways

- Specify first derivative at endpoints t_1 and t_n
- Force second derivative to be zero at endpoints, which gives *natural spline*
- Enforce "not-a-knot" condition, which forces two consecutive cubic pieces to be same
- Force first derivatives, as well as second derivatives, to match at endpoints t_1 and t_n (if spline is to be periodic)

Example: Cubic Spline Interpolation

- Determine natural cubic spline interpolating three data points (t_i, y_i), $i = 1, 2, 3$
- Required interpolant is piecewise cubic function defined by separate cubic polynomials in each of two intervals $[t_1, t_2]$ and $[t_2, t_3]$
- Denote these two polynomials by

$$p_1(t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2 + \alpha_4 t^3$$

$$p_2(t) = \beta_1 + \beta_2 t + \beta_3 t^2 + \beta_4 t^3$$

- Eight parameters are to be determined, so we need eight equations

Example, continued

- Requiring first cubic to interpolate data at end points of first interval $[t_1, t_2]$ gives two equations

$$\alpha_1 + \alpha_2 t_1 + \alpha_3 t_1^2 + \alpha_4 t_1^3 = y_1$$

$$\alpha_1 + \alpha_2 t_2 + \alpha_3 t_2^2 + \alpha_4 t_2^3 = y_2$$

- Requiring second cubic to interpolate data at end points of second interval $[t_2, t_3]$ gives two equations

$$\beta_1 + \beta_2 t_2 + \beta_3 t_2^2 + \beta_4 t_2^3 = y_2$$

$$\beta_1 + \beta_2 t_3 + \beta_3 t_3^2 + \beta_4 t_3^3 = y_3$$

- Requiring first derivative of interpolant to be continuous at t_2 gives equation

$$\alpha_2 + 2\alpha_3 t_2 + 3\alpha_4 t_2^2 = \beta_2 + 2\beta_3 t_2 + 3\beta_4 t_2^2$$

Example, continued

- Requiring second derivative of interpolant function to be continuous at t_2 gives equation

$$2\alpha_3 + 6\alpha_4 t_2 = 2\beta_3 + 6\beta_4 t_2$$

- Finally, by definition natural spline has second derivative equal to zero at endpoints, which gives two equations

$$2\alpha_3 + 6\alpha_4 t_1 = 0$$

$$2\beta_3 + 6\beta_4 t_3 = 0$$

- When particular data values are substituted for t_i and y_i, system of eight linear equations can be solved for eight unknown parameters α_i and β_i

Hermite Cubic vs Spline Interpolation

- Choice between Hermite cubic and spline interpolation depends on data to be fit and on purpose for doing interpolation
- If smoothness is of paramount importance, then spline interpolation may be most appropriate
- But Hermite cubic interpolant may have more pleasing visual appearance and allows flexibility to preserve monotonicity if original data are monotonic
- In any case, it is advisable to plot interpolant and data to help assess how well interpolating function captures behavior of original data

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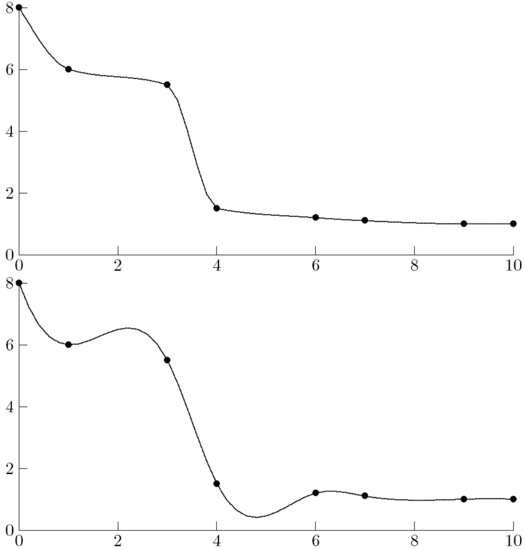
Hermite Cubic vs Spline Interpolation



B-splines

- *B-splines* form basis for family of spline functions of given degree
- B-splines can be defined in various ways, including recursion (which we will use), convolution, and divided differences
- Although in practice we use only finite set of knots $t_1, \dots, t_n$, for notational convenience we will assume infinite set of knots

  $$\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots$$

  Additional knots can be taken as arbitrarily defined points outside interval $[t_1, t_n]$
- We will also use linear functions

  $$v_i^k(t) = (t - t_i)/(t_{i+k} - t_i)$$

B-splines, continued

- To start recursion, define B-splines of degree 0 by

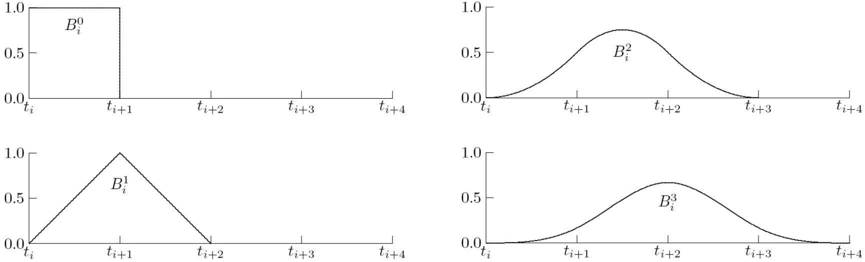
$$B_i^0(t) = \begin{cases} 1 & \text{if } t_i \le t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

  and then for $k > 0$ define B-splines of degree k by

$$B_i^k(t) = v_i^k(t) B_i^{k-1}(t) + (1 - v_{i+1}^k(t)) B_{i+1}^{k-1}(t)$$

- Since B_i^0 is piecewise constant and v_i^k is linear, B_i^1 is piecewise linear
- Similarly, B_i^2 is in turn piecewise quadratic, and in general, B_i^k is piecewise polynomial of degree k

B-splines, continued



< interactive example >

B-splines, continued

Important properties of B-spline functions B_i^k

1. For $t < t_i$ or $t > t_{i+k+1}$, $B_i^k(t) = 0$
2. For $t_i < t < t_{i+k+1}$, $B_i^k(t) > 0$
3. For all t, $\sum_{i=-\infty}^{\infty} B_i^k(t) = 1$
4. For $k \geq 1$, B_i^k has $k-1$ continuous derivatives
5. Set of functions $\{B_{1-k}^k, \ldots, B_{n-1}^k\}$ is linearly independent on interval $[t_1, t_n]$ and spans space of all splines of degree k having knots t_i

B-splines, continued

- Properties 1 and 2 together say that B-spline functions have local support
- Property 3 gives normalization
- Property 4 says that they are indeed splines
- Property 5 says that for given k, these functions form basis for set of all splines of degree k

B-splines, continued

- If we use B-spline basis, linear system to be solved for spline coefficients will be nonsingular and banded
- Use of B-spline basis yields efficient and stable methods for determining and evaluating spline interpolants, and many library routines for spline interpolation are based on this approach
- B-splines are also useful in many other contexts, such as numerical solution of differential equations, as we will see later