HW5: QR Iteration

- **Part 1:** Let $\bar{\mathbf{X}}_0 := \mathbf{A}$ and, for $i = 0, 1, 2, \ldots,$

$$
\begin{aligned}
&(1) \qquad \bar{\mathbf{Q}}_i \bar{\mathbf{R}}_i = \bar{\mathbf{X}}_i \\
&(2) \qquad \bar{\mathbf{X}}_{i+1} = \bar{\mathbf{R}}_i \bar{\mathbf{Q}}_i
\end{aligned}
$$

- Applying similarity transform involving $\bar{\mathbf{Q}}_i^T$ and $\bar{\mathbf{Q}}_i$ to (1) yields

$$
\begin{aligned}
\bar{\mathbf{Q}}_i^T (\bar{\mathbf{Q}}_i \bar{\mathbf{R}}_i) \bar{\mathbf{Q}}_i &= \bar{\mathbf{Q}}_i^T \bar{\mathbf{X}}_i \bar{\mathbf{Q}}_i \\
\bar{\mathbf{R}}_i \bar{\mathbf{Q}}_i &= \bar{\mathbf{Q}}_i^T \bar{\mathbf{X}}_i \bar{\mathbf{Q}}_i \\
\bar{\mathbf{X}}_{i+1} &= \bar{\mathbf{Q}}_i^T \bar{\mathbf{X}}_i \bar{\mathbf{Q}}_i,
\end{aligned}
$$

  which, with $\bar{\mathbf{X}}_0 = \mathbf{A}$, implies that $\bar{\mathbf{X}}_i = (\bar{\mathbf{Q}}_{i-1}^T \ldots \bar{\mathbf{Q}}_0^T) \mathbf{A} (\bar{\mathbf{Q}}_0 \ldots \bar{\mathbf{Q}}_{i-1})$.
  (You should be slightly more verbose here.)

- **Part 2:** Note that $\mathbf{A} = \bar{\mathbf{Q}}_0\bar{\mathbf{R}}_0$ and

$$\begin{aligned} \mathbf{A}^2 &= \bar{\mathbf{Q}}_0\bar{\mathbf{R}}_0\bar{\mathbf{Q}}_0\bar{\mathbf{R}}_0 \\ &= \bar{\mathbf{Q}}_0\bar{\mathbf{Q}}_1\bar{\mathbf{R}}_1\bar{\mathbf{R}}_0, \end{aligned}$$

  where the relationship $\bar{\mathbf{Q}}_1\bar{\mathbf{R}}_1 = \bar{\mathbf{X}}_1 = \bar{\mathbf{R}}_0\bar{\mathbf{Q}}_0$ follows from applying Step 1 above with $i = 1$ and Step 2 with $i = 0$.

- For the next power, we have

$$\begin{aligned} \mathbf{A}^3 &= (\bar{\mathbf{Q}}_0\bar{\mathbf{Q}}_1\bar{\mathbf{R}}_1\bar{\mathbf{R}}_0)(\bar{\mathbf{Q}}_0\bar{\mathbf{R}}_0) \\ &= (\bar{\mathbf{Q}}_0\bar{\mathbf{Q}}_1\bar{\mathbf{R}}_1\bar{\mathbf{Q}}_1\bar{\mathbf{R}}_1\bar{\mathbf{R}}_0) \\ &= (\bar{\mathbf{Q}}_0\bar{\mathbf{Q}}_1\bar{\mathbf{Q}}_2\bar{\mathbf{R}}_2\bar{\mathbf{R}}_1\bar{\mathbf{R}}_0) \end{aligned}$$

- Every time we introduce a new multiple of $\mathbf{A} = \bar{\mathbf{Q}}_0\bar{\mathbf{R}}_0$ on the right, the $\bar{\mathbf{Q}}_0$ will swap with the existing rightmost $\bar{\mathbf{R}}_0$ in the preceding power to yield $\bar{\mathbf{Q}}_1\bar{\mathbf{R}}_1$. Then the adjacent $\bar{\mathbf{R}}_1\bar{\mathbf{Q}}_1$ will be promoted to $\bar{\mathbf{Q}}_2\bar{\mathbf{R}}_2$, etc., such that $\bar{\mathbf{Q}}_j$ propagates to the left with j being incremented until $j = k - 1$ in the expression for $\mathbf{A}^k$.

- The final result is $$\mathbf{A}^k = (\underbrace{\bar{\mathbf{Q}}_0\bar{\mathbf{Q}}_1 \ldots \bar{\mathbf{Q}}_{k-1}}_{\mathbf{Q}_{k-1}}\underbrace{\bar{\mathbf{R}}_{k-1} \ldots \bar{\mathbf{R}}_1\bar{\mathbf{R}}_0}_{\mathbf{R}_{k-1}}).$$

- Note that $\mathbf{Q}_{k-1}$ is a product of orthogonal matrices and, hence, orthogonal, while $\mathbf{R}_{k-1}$ is a product of upper triangular matrices and thus also upper triangular, so the above expression is a QR factorization of $\mathbf{A}^k$.

- Up to the sign of the columns of $\mathbf{Q}$, the QR factorization of $\mathbf{A}^k$ is unique—the first column of $\mathbf{Q}_{k-1}$ is the normalized first column of $\mathbf{A}^k$, the second column of $\mathbf{Q}_{k-1}$ is the second column of $\mathbf{A}^k$, minus its projection onto $\mathbf{q}_1$, and normalized, etc.

- **Part 3:** Here, start with orthogonal iteration. Let $\mathbf{X}_0 := \mathbf{A}$ and, for $i = 0, 1, 2, \ldots,$

$$\begin{aligned}
&(1) \quad \mathbf{Q}_i \mathbf{R}_i = \mathbf{X}_i \\
&(2) \quad \mathbf{X}_{i+1} = \mathbf{A}\mathbf{Q}_i \\
&(3) \quad \hat{\mathbf{X}}_{i+1} = \mathbf{Q}_i^T \mathbf{A} \mathbf{Q}_i = \mathbf{Q}_i^T \mathbf{X}_{i+1}
\end{aligned}$$

  where, here, the $\mathbf{Q}_i$s and $\mathbf{R}_i$ are *not* the same as in Parts 1 and 2.

- Note that (1) and (2) imply

$$\begin{aligned}
\mathbf{Q}_0 &= \mathbf{X}_0 \mathbf{R}_0^{-1} = \mathbf{A}\mathbf{R}_0^{-1} \\
\mathbf{X}_1 &= \mathbf{A}\mathbf{Q}_0 = \mathbf{A}^2 \mathbf{R}_0^{-1} \\
\mathbf{Q}_1 &= \mathbf{X}_1 \mathbf{R}_1^{-1} = \mathbf{A}^2 \mathbf{R}_0^{-1} \mathbf{R}_1^{-1} \\
&\vdots \\
\mathbf{Q}_{k-1} &= \mathbf{A}^k \mathbf{R}_0^{-1} \mathbf{R}_1^{-1} \cdots \mathbf{R}_{k-1}^{-1}
\end{aligned}$$

- Now multiply both expressions from the right by $\mathbf{R}_j$, for $j = k-1$, $k-2$,..., 1, 0 to obtain a QR factorization of $\mathbf{A}^k$.

- Equate this result to the $\mathbf{A}^k$ of Part 2 to establish the equivalence of the Q factors. From this, the equivalence of the requested $\mathbf{X}$ matrices follows in a straightforward way.