# Numerical Methods for Partial Differential Equations

CS555 / MATH 555 / CSE 510

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# Outline

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# What's the point of this class?

PDEs describe lots of things in nature:

- Fluid flow
- Electromagnetics
- Adv / diff
- Plasmas
- Mechanics

Idea: Use them to

- make predictions and check them / check models
- use predictions to answer design questions

# Survey

- Home dept
- Degree pursued
- Longest program ever written
  - in Python?
- Research area

# Class web page

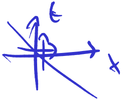
https://bit.ly/numpde-s20

- Book Draft
- Notes, Class Outline
- Assignments (submission and return)
- Piazza
- Grading Policies/Syllabus
- Video
- Scribbles
- Demos (binder)

# Outline

# PDEs: Example I

$u(x,t)$




What does this do? $\partial_t u = \partial_x u$

- slope in x and t is identical
- maybe a profile propagated on a diag

# PDEs: Example II

"bendiness in x"

What does this do? $\partial_x^2 u + \partial_y^2 u = 0$

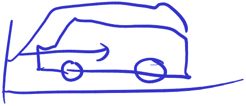
"bendiness in x" = − "b. in y"

- maybe no interior maxima

# Some good questions

causality

- What is a time-like variable? (Variables labeled $t$?)
- What if there are boundaries?
  - In space? "BC"
  - In time? "IC"



- Existence and Uniqueness of Solutions?
  - Depends on where we look (the *function space*)
  - In the case of the two examples? (if there are no boundaries?)

Some general takeaways:

use intuition, it helps

# PDEs: An Unhelpfully Broad Problem Statement

Looking for $u : \Omega \to R^n$ where $\Omega \subseteq \mathbb{R}^d$ so that $u \in V$ and

$$F(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots, x, y, \ldots) = 0$$

## Notation

Used as convenient:

$$u_x = \partial_x u = \frac{\partial u}{\partial x}$$

# Properties of PDEs

What is the order of the PDE?

highest derivative taken

When is the PDE linear?

have $u, v$ solutions $\alpha u + \beta v$ also a sol.

When is the PDE quasilinear?

The dep. in $F$ on the highest-order deriv. is linear in $u$.

When is the PDE semilinear?

It is quasilinear and if the h.o.c. only depend on space.

# Examples: Order, Linearity?

$$(xu^2)u_{xx} + (u_x + y)u_{yy} + u_x^3 + yu_y = f$$

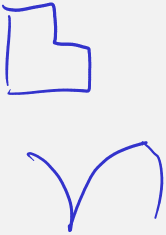
QL 2nd

$$(x + y + z)u_x + (z^2)u_y + (\sin x)u_z = f$$

SL 1st

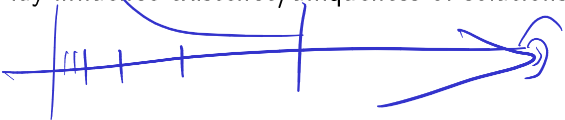
# Properties of Domains

$\Omega \subseteq \mathbb{R}^d$



- has corner
- convex / has reentrant corner
- satisfies minimum angle reg.

May influence existence/uniqueness of solutions!



# Function Spaces: Examples

Name some function spaces with their norms.

- $C^0$ : continuous
- $C^1$
- $C_L$ : Lipschitz continuous
- $L_p(\Omega)$ $\|f\|_p = \sqrt[p]{\int_\Omega |f(x)|^p dx}$
  - $\hookrightarrow p=2$: inner product / define equiv. classes
- $W_p^1(\Omega)$ : $\|f\|_{W_p^1} = \|f\|_p + \|f'\|_p < \infty$

May *also* influence existence/uniqueness of solutions!

$H^1 = W_2^1$ Hilbert: complete + inner prod.

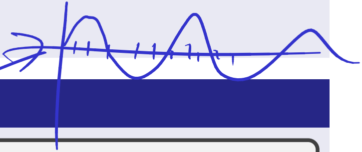
# Solving PDEs

Closed-form solutions:

- If separation of variables applies to the domain: good luck with your ODE
- If not: Good luck! $\rightarrow$ Numerics

## General Idea (that we will follow some of the time)

- Pick $V_h \subseteq V$ finite-dimensional
  - $h$ is often a *mesh spacing*
- Approximate $u$ through $u_h \in V_h$
- Show: $u_h \rightarrow u$ (in some sense) as $h \rightarrow 0$



## Example

# About grand big unifying theories

Is there a grand big unifying theory of PDEs?

no

# Collect some stamps

$$a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}+d(x,y)u_x+e(x,y)u_y+f(x,y)u=g(x,y)$$

| Discriminant value | Kind | Example |
|---|---|---|
| $b^2-4ac<0$ | Elliptic | Laplace $u_{xx}+u_{yy}=0$ |
| $b^2-4ac=0$ | Parabolic | Heat $u_t=u_{xx}$ |
| $b^2-4ac>0$ | Hyperbolic | Wave $u_{tt}=u_{xx}$ |

Where do these names come from?

quadratic forms

# Classification in higher dimensions

$$Lu := \sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \text{lower order terms}$$

Consider the matrix $A(x) = (a_{ij}(x))_{i,j}$. May assume $A$ symmetric. Why?

Schwarz's thm.

What cases can arise for the eigenvalues?

real eigvals

$\lambda = 0$ for some $\lambda$ $\longrightarrow$ parabolic

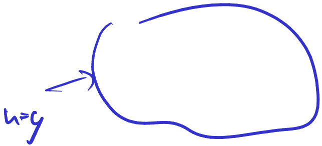
$\lambda(x)$ all have same sign $\longrightarrow$ elliptic

$\lambda(x)$ all but one $\rightarrow$ hyperbolic

more than one of each sign

$\longrightarrow$ ultra-hyperbolic

# Elliptic PDE: Laplace/Poisson Equation



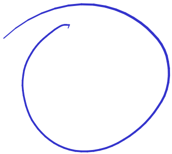

$$\triangle u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \nabla \cdot \nabla u(x) \overset{\text{2D}}{=} u_{xx} + u_{yy} = f(x) \quad (x \in \Omega)$$

Called Laplace equation if $f = 0$. With Dirichlet boundary condition

$$u(x) = g(x) \qquad (x \in \partial\Omega).$$

**Demo:** Elliptic PDE Illustrating the Maximum Principle

$g(x) = \sin(\alpha)$



$u(r,\theta) = U(r) \cdot V(\theta)$

# Elliptic PDEs: Singular Solution

**Demo:** Elliptic PDE Radially Symmetric Singular Solution

Given $G(x) = C\log(|x|)$ as the free-space Green's function, can we construct the solution to the PDE with a more general $f$?

What can we learn from this?