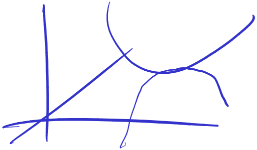
Today	Announcements
- classification	- Office Hours
- computing derivatives	- HW1
- FD for hyperbolic	

Elliptic PDE: Laplace/Poisson Equation



$$\triangle u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \nabla \cdot \nabla u(x) \overset{\text{2D}}{=} u_{xx} + u_{yy} = f(x) \quad (x \in \Omega)$$

Called Laplace equation if $f = 0$. With Dirichlet boundary condition

$$u(x) = g(x) \qquad (x \in \partial\Omega).$$

Demo: Elliptic PDE Illustrating the Maximum Principle

Elliptic PDEs: Singular Solution

$$\int \Delta w(x)\,\varphi(x) = \int \delta(x)\,\varphi(x)\,dx = \varphi(0)$$

Demo: Elliptic PDE Radially Symmetric Singular Solution

Given $G(x) = C\log(|x|)$ as the free-space Green's function, can we construct the solution to the PDE with a more general f?

$$u(x) = G * f(x) = \int G(x-y) f(y)\,dy$$

What can we learn from this?

solutions to elliptic PDEs have global dependence on input data

$$\Delta u = f \qquad \Delta G = \delta$$

Elliptic PDEs: Justifying the Singular Solution

$(f * g)' = (f' * g) = (f * g')$

$$u(x) = (G * f)(x) = \int_{\mathbb{R}^d} G(x-y)f(y)dy$$

Why?

$\Delta u(x) = \Delta(G * f)(x) = (\Delta G * f)(x) = \int \delta(x-y) f(y)\, dy = f(x)$

Parabolic PDE: Heat Equation

$$v'(t) \cdot w(x) = v(t) \cdot w''(x)$$

$$\frac{v'(t)}{v(t)} = c = \frac{w''(x)}{w(x)}$$

$$u_t = u_{xx} \qquad ((x,t) \in [0,1] \times [0,T])$$
$$u(x,0) = g(x) \qquad (x \in [0,1])$$
$$u(0,t) = u(1,t) = 0 \qquad (t \in [0,T])$$

with $g(x) = \sin(\pi x)$. Separation-of-variables analytic solution?

$$u(x,t) = v(t) \cdot w(x)$$
$$w''(x) = c\,w(x) \qquad w(0) = w(1) = 0 \;\Big|\; w(x) = \alpha \sin(n\pi x)$$
$$v'(t) = -n^2\pi^2 v(t)$$
$$c = -n^2\pi^2$$
$$v(t) = e^{-n^2\pi^2 t}$$
$$u(x,t) = \alpha \cdot \sin(n\pi x) \cdot e^{-n^2\pi^2 t}$$
$$\text{IC} \Rightarrow n = 1 \quad \alpha = 1$$

Parabolic PDE: Solution Behavior

Demo: Parabolic PDE What can we learn from analytic and numerical solution?

- washes out solution
- appears to satisfy a maximum principle

- appears to smooth the input data

Hyperbolic PDE: Wave Equation

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} \qquad ((x,t) \in \mathbb{R} \times [0, T]) \\
u(x,0) &= g(x) \qquad (x \in \mathbb{R})
\end{aligned}
$$

with $g(x) = \sin(\pi x)$.

Is this problem well-posed?

$u_t(x,0) = 0 \qquad (x \in \mathbb{R})$

Can be rewritten in conservation law form:

$\vec{q}_t + \nabla \cdot F(q) = s(x)$

Hyperbolic Conservation Laws

$q \in \mathbb{R}^n$ $\qquad$ $x \in \mathbb{R}^d$ $\qquad$ $\operatorname{div} F > 0$

$$q_t(x,t) + \nabla \cdot \vec{F}(q(x,t)) = \overset{0}{\cancel{s(x)}}$$

Why is this called a conservation law?

- Balance of conserved quantity q and a flux F
- Flux function F prescribes is direction of flow
- Integral form shows "flow across boundary"

F :? →?

$F : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^d$

Wave Equation as a Conservation Law

$$u_{tt} = c^2 u_{xx}$$

Rewrite the wave equation in conservation law form: $q_t + \nabla \cdot F(q) = 0$

Introduce a new variable v

$$u_t = c v_x$$

$$v_t = c u_x$$

$$u_{tt} = c v_{xt} = c^2 u_{xx}.$$

Solving Conservation Laws

Solve

$$u_{tt} = c^2 u_{xx}$$

$$q = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$u_t = c v_x$$
$$v_t = c u_x.$$

$$q_t + \nabla \cdot \underbrace{\begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{q} = 0$$

eigenvalues: $c, -c$

$$\tilde{q} := V^{-1} q$$

$$\tilde{q} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

$$\tilde{q}_t + \underbrace{V^{-1} A V}_{\begin{pmatrix} c & \\ & -c \end{pmatrix}} \tilde{q}_x = 0$$

$$\tilde{u}_t = -c \tilde{u}_x$$

$$\tilde{v}_t = c \tilde{v}_x$$

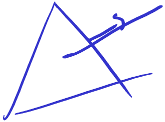
Solution: $u(x,t) = \phi_l(x+ct) + \phi_r(x-ct)$

Demo: Hyperbolic PDE

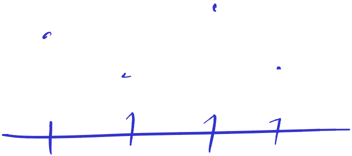
Hyperbolic: Solution Properties

Properties of the solution for hyperbolic equations:

- models conserved quantities
- energies (conserved) (→HW1)
- Maintain the smoothness of the IC
- characteristic decomposition: popular trick



Outline



Interpolation and Vandermonde Matrices

$V\vec{\alpha} =$ point values

$$p(x_i) = \sum_{j=0} \alpha_j \underbrace{\varphi_j(x_i)}_{V_{ij}}$$

generalized Vandermonde matrix

$$V\vec{\alpha} = p(\vec{x})$$

$$p'(x_i) \approx \sum_{j=0} \alpha_j \underbrace{\varphi'_j(x_i)}_{V'_{ij}} \rightsquigarrow V'\vec{\alpha} \approx p'(\vec{x})$$

$$p'(\vec{x}) \approx \underbrace{V' \quad V^{-1}}\, p(\vec{x})$$

n points:

Interpolation: h^n

Differentiation: h^{n-1}

Taking Derivatives Numerically

Why *shouldn't* you take derivatives numerically?

Demo: Taking Derivatives with Vandermonde Matrices