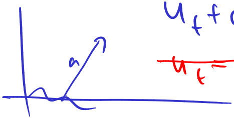
Today

- 1D advection
  - ↳ schemes
  - ↳ theory

Announcements

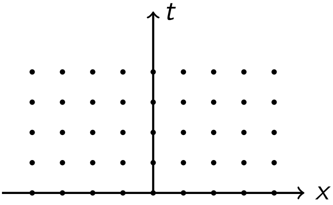
- HW1
- HW2
- next week

Finite Difference for Hyperbolic: Idea



$u_t + a u_x = 0$ ✓

$\cancel{u_t = a u_x}$



$$\{(x_k, t_\ell) : x_k = k h_x, t_\ell = \ell h_t\}$$

If $u(x,t)$ is the exact solution, want

$$u_{k,\ell} \approx u(x_k, t_\ell).$$

Condition at each grid point?

- replace derivatives w/ stencils
- get system of eqns
- solve

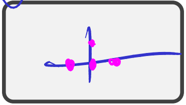
What are explicit/implicit schemes?

explicit: update formula

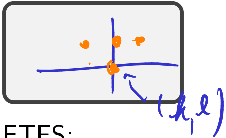
implicit: system solve

Designing Stencils

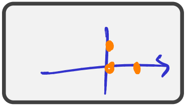
ETCS:



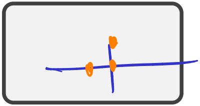
ITCS:



ETFS:



ETBS:



Terminology?

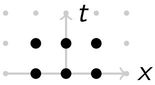
Explicit / Implicit Time

Centered / Forward / Backward

Write out ITCS:

$$\frac{u_{k,l+1}-u_{k,l}}{h_t} + a\,\frac{u_{k+1,l+1}-u_{k-1,l+1}}{2h_x}$$

Crank-Nicolson



Crank-Nicolson

Write out Crank-Nicolson:

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + \frac{a}{2}\left(\frac{u_{k+1,\ell+1} - u_{k-1,\ell+1}}{2h_x} + \frac{u_{k+1,\ell} - u_{k-1,\ell}}{2h_x} \right).$$

Lax-Wendroff

What's the core idea behind Lax-Wendroff?

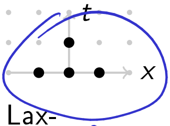
- Taylor series in time
- Trade time derivs for space

= FTCS

Write out Lax-Wendroff.

$u_t = -a u_x, \quad u_{tt} = -a(u_x)_t = -a(u_t)_x = a^2 u_{xx}$

$$u_{k,\ell+1} - u_{k,\ell}$$
$$\approx h_t u_t(x_k, t_\ell) + \frac{h_t^2}{2} u_{tt}(x_k, t_\ell)$$
$$= -a h_t u_x(x_k, t_\ell) + a^2 \frac{h_t^2}{2} u_{xx}(x_k, t_\ell)$$
$$\approx -a h_t \frac{u_{k+1,\ell} - u_{k-1,\ell}}{2h_x} + \frac{a^2 h_t^2}{2} \cdot \frac{u_{k+1,\ell} - 2u_{k,\ell} + u_{k-1,\ell}}{h_x^2}$$



Lax-Wendroff

pics aren't unique

Exploring Advection Schemes

Demo: Methods for 1D Advection

- Which of the schemes "work"?
- Any restrictions worth noting?

Outline

A Matrix View of Two-Level Stencil Schemes

Define

$$\boldsymbol{v}_\ell = \begin{bmatrix} u_{1,\ell} \\ \vdots \\ u_{N_x,\ell} \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_{N_t} \end{bmatrix}.$$

$$\frac{v_{\ell+1} - v_\ell}{h_t}$$

Define

$$\boldsymbol{u}_\ell = \begin{bmatrix} u(x_1, t_\ell) \\ \vdots \\ u(x_{N_x}, t_\ell) \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_{N_t} \end{bmatrix}.$$

Definition (Two-Level Finite Difference Scheme)

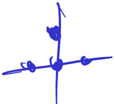
A finite difference scheme that can be written as

$$P_h v_{\ell+1} = Q_h v_\ell + \underline{h_t} b_\ell$$

is called a two-level linear finite difference scheme.

- $b_\ell = 0$ mostly
- may also consider inf. grids

Rewriting Schemes in Matrix Form



$$P_h \boldsymbol{v}_{\ell+1} = Q_h \boldsymbol{v}_\ell + h_t \boldsymbol{b}_\ell$$

Find P_h and Q_h for ETCS:

$P_h = I$

$Q_h = \text{tridiag}\left(-\frac{ah_t}{2h_x}, 1, \frac{ah_t}{2h_x}\right)$

Find P_h and Q_h for Crank-Nicolson:

$P_h = \text{tridiag}\left(-\frac{ah_t}{4h_x}, 1, \frac{ah_t}{4h_x}\right)$

$Q_h = \text{tridiag}\left(-\frac{ah_t}{4h_x}, 1, \frac{ah_t}{4h_x}\right)$

Truncation Error

Definition (Truncation Error)

The local truncation error $\tau_{u,\ell}$ is the error that remains if we plug an exact solution into FD scheme.

Demo: Truncation Error Analysis via sympy

Error and Error Propagation

Express truncation error in our two-level framework:

$$P_h u_{\ell+1} = Q_h u_\ell + h_t \tau_\ell$$

Define $\boldsymbol{e}_\ell = \boldsymbol{u}_\ell - \boldsymbol{v}_l$. Understand the error as accumulation of truncation error:

$$P_h v_{\ell+1} = Q_h v_\ell$$

$$e_0 = 0 \qquad P_h e_{\ell+1} = Q_h e_\ell + h_t \tau_\ell$$

$$e_{\ell+1} = P_h^{-1} Q_h e_\ell + h_t P_h^{-1} \tau_\ell$$

Discrete and Continuous Norms

To measure properties of numerical solutions we need norms. Define a discrete L^∞ norm.

$$\|e\|_\infty = \max_{k,l} |e_{k,l}|$$

Define a discrete L^2 norm.

$$\|e\|_2 = \sqrt{\sum_{k,l} |e_{k,l}|^2 h_x h_t} \approx \iint |e|^2$$

Important features:

- should retain same scaling as h_x, h_t change (approx. continuous norm)

Consistency and Convergence

Assume $u, (\partial_x^{q_x})u, (\partial_t^{q_t})u \in L^2(\mathbb{R} \times [0, t^*])$.

Definition (Consistency)

A two-level scheme is consistent in the L^2-norm with order q_t in time and q_x in space if

$$\boxed{\max_{\ell} \|\tau_\ell\|_2 = O(h_x^{q_x} + h_t^{q_t})}$$

Definition (Convergence)

A two-level scheme is convergent in the L^2-norm with order q_t in time and q_x in space if

$$\boxed{\max_{\ell} \|e_\ell\|_2 = O(h_x^{q_x} + h_t^{q_t})}$$

Analyzing ETFS

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\frac{u_{k+1,\ell} - u_{k,\ell}}{h_x} = 0$$

Let's understand more precisely what happens for this scheme.