Today

Announcements

- HW2 due ~~Fri~~ Wed

Toeplitz operators

$$(Tu)_k = \sum_j u_j t_{k-j}$$

$$(u * v)(x) = \int u(y)\, v(x-y)\, dy$$

$$u_t + a u_x = 0$$

ETBS

CFL

$$\lambda = \frac{a h_t}{h_x}$$

$$0 \le \lambda \le 1$$

von Neumann Stability

Two-level finite difference scheme

$$P_h \boldsymbol{v}_{\ell+1} = Q_h \boldsymbol{v}_\ell + h_t \boldsymbol{b}_\ell,$$

where P_h and Q_h are Toeplitz operators with vectors $\boldsymbol{p}$ and $\boldsymbol{q}$.

Definition (Symbol of a Two-Level Finite Difference Scheme)

Let

$$\hat{\boldsymbol{p}}(\theta) = \sum_k p_k e^{-i\varphi k}, \qquad \hat{\boldsymbol{q}}(\theta) = \sum_k q_k e^{-i\varphi k}.$$

Then the symbol of the two-level FD method is $s(\varphi) = \hat{\boldsymbol{q}}(\varphi)/\hat{\boldsymbol{p}}(\theta)$.

Definition (Von Neumann Stability)

If

$$\max_\varphi |s(\varphi)| \leq 1, \qquad \max_\varphi \left| \frac{1}{\hat{\boldsymbol{p}}(\theta)} \right| \leq c$$

for some constant $c > 0$, we say the scheme is von Neumann stable.

- Examples of vN stability
  - Dispersion / dissipation
    - ↳ parabolic
    - ↳ Scalar cons. laws / FV methods

Comparison with Lax-Richtmyer Stability

$a > 0$

$u_t + a u_x$

$u(x,0) = g(x) \quad x \in [0,1]$

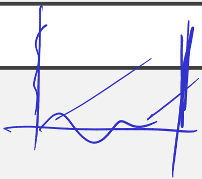
$u(1,t) = 5$

Need $\|(P_h^{-1}Q_h)^\ell P_h^{-1}\| \le c.$

$\|(P_h^{-1}Q_h)^\ell\| \le 1 \qquad \|P_h^{-1}\| \le c$

Why is bounding the symbol the most salient part?

$\left|\frac{\hat{q}(\varphi)}{\hat{p}(\varphi)}\right| \le 1$ as a necessary cond.



Main restriction of von Neumann stability?

Restricted to inf / periodic grid

von Neumann Stability: ETBS (1/2)

ETBS: Let $\lambda = ah_t/h_x$. $u_{k,\ell+1} = \lambda u_{k-1,\ell} + (1-\lambda)u_{k,\ell}$.

$P_h = I \qquad Q_h = \operatorname{tridiag}(\lambda, 1-\lambda, 0)$

Fourier transform $r_k = \delta_{k,j}$

$$\hat{r}(\varphi) = \sum_k r_k e^{-i\varphi k} = \sum_k \delta_{k,j} e^{-i\varphi k} = e^{-ij\varphi}$$

$$\hat{p}(\varphi) = 1, \quad \hat{q}(\varphi) = (1-\lambda) + \lambda e^{-i\varphi} = 1 - \lambda(1 - e^{-i\varphi})$$

$$|s(\varphi)|^2 = \hat{q}(\varphi)\hat{q}^*(\varphi)$$

$$= 2\lambda(1-\lambda)\cos(\varphi) + 2\lambda^2 - 2\lambda + 1$$

von Neumann Stability: ETBS (2/2)

Found: $|s(\varphi)|^2 = 1 + 2(\lambda - \lambda^2)(\cos\varphi - 1)$.

$\frac{d}{d\varphi}|s(\varphi)|^2 = -2(\lambda - \lambda^2)\sin(\varphi) = 0$ whenever $\varphi \in \pi\mathbb{Z}$. There

$s(m\pi) = 1 + 2(\lambda - \lambda^2)((-1)^m - 1)$

For m even, $s = 0$. For m odd,

$s(m\pi) = 1 + 2(\lambda - \lambda^2)(-2) = (1 - 2\lambda)^2$

$|1 - 2\lambda| \leq 1 \Leftrightarrow 0 \leq \lambda \leq 1 \Leftrightarrow 0 \leq h_t \leq \frac{h_x}{a}$.

von Neumann Stability: ETCS

Let $\lambda = ah_t/h_x$. Then

$$u_{k,\ell+1} = \frac{\lambda}{2}u_{k-1,\ell} + u_{k,\ell} - \frac{\lambda}{2}u_{k+1,\ell}.$$

$P_h = I \quad , \quad Q_h = \mathrm{tridiag}\left(\frac{\lambda}{2}, 1, -\frac{\lambda}{2}\right)$

$$\hat{q}(\varphi) = \frac{\lambda}{2}e^{-i\varphi} + 1 - \frac{\lambda}{2}e^{-i\varphi(-1)} = 1 - \lambda \sin(\varphi);$$

$$|s(\varphi)|^2 = |\hat{q}(\varphi)|^2 \geq 1.$$

von Neumann Stability: Crank-Nicolson

Let $\lambda = ah_t/(4h_x)$

$$-\lambda u_{k-1,\ell+1} + u_{k,\ell+1} + \lambda u_{k+1,\ell+1} = +\lambda u_{k-1,\ell} + u_{k,\ell} \cancel{+} - \lambda u_{k+1,\ell}.$$

$P_h = \text{tridiag}(-\lambda, 1, \lambda)$ $\quad$ $Q_h = \text{tridiag}(\lambda, 1, -\lambda)$

$$\hat{p}(\varphi) = -\lambda e^{-i\varphi} + 1 + \lambda e^{i\varphi} = 1 + 2\lambda i \sin(\varphi)$$

$$\hat{q}(\varphi) = 1 - 2\lambda i \sin(\varphi)$$

$$s(\varphi) = \frac{\hat{q}(\varphi)\,\hat{q}(\varphi)^*}{\hat{p}(\varphi)\,\hat{p}(\varphi)^*} = 1$$

Outline

Studying Solutions of the PDE

$u_t = u_{xx}$

$u_t = -u_{xx}$

Saw numerically: interesting dispersion/dissipation behavior.
Want: theoretical understanding.

Consider *linear*, *continuous* (not yet discrete) differential operators

$$
\begin{aligned}
L_1 u &= u_t + a u_x, \\
L_2 u &= u_t - D u_{xx} + a u_x, \qquad (D > 0) \\
L_3 u &= u_t + a u_x - \mu u_{xxx}.
\end{aligned}
$$

What could we use as a prototype solution?

A Prototype Solution of the PDE

Observation: all these operators are diagonalized by complex exponentials. Come up with a 'prototype complex exponetial solution'.

$$z(x,t) = e^{i(kx - \omega t)}$$

What type of function is this?

k, ω real: traveling wave with speed $c = \frac{\omega}{k}$

k imaginary: evanescent wave

$\operatorname{Im} \omega < 0$: decay in time

Wave-like Solutions of the PDE

$L_1 = u_t + a u_x$

$L_1 u = 0$

$$z(x, t) = z_0 e^{i(kx - \omega t)}$$

Observations in connection with L_1?

$Lz = \lambda(\omega, k) z$

$z(x,t)$ is a solution of $L \iff \lambda(\omega, k) = 0$

What is the dispersion relation?

$\lambda(\omega, k) = 0$

Picking Apart the Dispersion Relation

Consider $\omega(k) = \alpha(k) + i\beta(k)$. Rewrite the wave solution with this.

$$z(x,t) = z_0 e^{i(kx - \omega t)}$$

$$= z_0 e^{i(kx - \alpha(k)t - i\beta(k)t)}$$

$$= z_0 e^{\beta(k)t} e^{i(kx - \alpha(k)t)}$$

How can we recognize dissipation?

$\beta(k) < 0 \rightarrow$ PDE dissipative

What is the phase speed? How can we recognize dispersion?

$$v_{ph} = \frac{\alpha(k)}{k}$$

If v_{ph} is a constant $\Leftrightarrow \alpha(k)$ linear in k, then the PDE is not dispersive. (Otherwise ...)

Dispersion Relation: Examples

$x_0 e^{i(kx-\omega t)}$

In each case, find the dispersion relation and identify properties.

$L_1 u = u_t + a u_x$

$0 = \lambda(\omega, k) = i(ak - \omega) \Rightarrow \omega = ak$

no dissipation, no dispersion

$L_2 u = u_t - D u_{xx} + a u_x$

$0 = \lambda(\omega, k) = -i\omega + iak + Dk^2 \Rightarrow \omega = ak - iDk^2$

dissipative, no dispersion

$L_3 u = u_t + a u_x - \mu u_{xxx}$

$\omega = ak + \mu k^3$

not dissipative, dispersive

Outline

Numerical Dissipation/Dispersion Analysis

Goal: Want discrete finite difference scheme to match dissipation/dispersion behavior of continuous PDE.

Define a discrete wave-like function:

$$z_{j,\ell} = e^{i(k \underbrace{j h_x}_{x} - \omega \underbrace{\ell h_t}_{t})}$$

We want $\boldsymbol{z}$ to solve $P_h \boldsymbol{z}_{\ell+1} = Q_h \boldsymbol{z}_\ell$. How can we connect the operators to the wave solution?

View P_h, Q_h as Toeplitz.