Today

- Num disp./diss.
- Parabolic
- Theory on cons. laws

Announcements

HW2 due Wed

Numerical Dissipation/Dispersion Analysis

$e^{i(kx-\omega t)}$

$\omega(k)$

Goal: Want discrete finite difference scheme to match dissipation/dispersion behavior of continuous PDE.

Define a discrete wave-like function:

$$z_{k,\ell} = z_0 e^{i(k h_x - \omega \ell h_t)}$$

We want $\boldsymbol{z}$ to solve $P_h \boldsymbol{z}_{\ell+1} = Q_h \boldsymbol{z}_\ell$. How can we connect the operators to the wave solution?

Toeplitz

Toeplitz and Waves

$$z_{j,\ell} = z_0 e^{i(kjh_x - \omega \ell h_t)}.$$

Theorem (Waves Diagonalize Toeplitz Operators)

Let T be a Toeplitz operator. Then $T\boldsymbol{z}_\ell = \lambda(k)\boldsymbol{z}_\ell = \hat{\boldsymbol{t}}(kh_x)\boldsymbol{z}_\ell$.

$$(Tz)_j = \sum_m z_{m,\ell}\, t_{j-m} = \sum e^{i(kmh_x - \omega h_t \ell)}\, t_{j-m}$$

$$= \sum_m e^{i(k(m-j)h_x)}\, t_{j-m}\, e^{i(kjh_x - \omega \ell h_t)}$$

$$m' = j-m$$

$$= \left(\sum_{m'} e^{-ikm'h_x} t_{m'}\right) e^{i(kjh_x - \omega \ell h_t)}$$

$$= \hat{t}(kh_x)\, z_{j,\ell}$$

Waves and Two-Level Schemes

Since P_h and Q_h are Toeplitz, we must have

$$P_h \mathbf{z}_{\ell+1} = \lambda_P(k)\mathbf{z}_{\ell+1}, \qquad Q_h \mathbf{z}_\ell = \lambda_Q(k)\mathbf{z}_\ell.$$

What does that mean?

$$\lambda_P(k) z_{\ell+1} = \lambda_Q(k) z_\ell$$

$$\lambda_P(k) e^{i(kjh_x-(\ell+1)h_t\omega)} = \lambda_Q(k) e^{i(kjh_x-\ell h_t\omega)} \quad (\forall j)$$

$$\to \quad e^{-i\omega h_t} = \frac{\lambda_Q(k)}{\lambda_P(k)} = \frac{\hat{q}(kh_x)}{\hat{p}(kh_x)} = s(kh_x)$$

Seen before?

Symbol, vN.

Discrete Dispersion Relation (1/2)

$e^{i(jkh_x - \omega \ell h_t)}$

So $\boldsymbol{z}_\ell$ is a solution of the finite difference scheme if $\omega = \omega(kh_x)$ satisfies

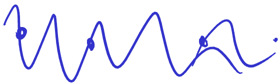
$$e^{-i\omega(\kappa)h_t} = s(\kappa), \qquad e^{i(kx - \omega t)}$$

where we let $\kappa = kh_x$. Interpret κ. $\sim \frac{k}{N}$

wavelengths per point.

Let $s(\kappa) = |s(\kappa)|\, e^{i\varphi(\kappa)} = e^{\log|s(\kappa)| + i\varphi(\kappa)}$. $\omega(\kappa)$?

$$\omega(\kappa) = \frac{-\varphi(\kappa) + i \log|s(\kappa)|}{h_t}$$



Discrete Dispersion Relation (2/2)

$$\omega(\kappa) = \frac{-\varphi(\kappa) + i \log |s(\kappa)|}{h_t}.$$

Plug that into the wave-like solution:

$$z_{j,\ell} = e^{i(kjh_x - \omega(\kappa)\ell h_t)}$$

$$= |s(\kappa)|^{\ell} e^{\underbrace{ik(jh_x - \frac{-\varphi(\kappa)}{kh_t}\ell h_t)}}$$

Criterion for stability?

$$|s(k)| \leq 1$$

Numerical Dispersion/Dissipation

Finite difference scheme $P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell$ with symbol $s(k)$.

$$z_{j,\ell} = z_0 \, e^{\log|s(\kappa)|\ell} e^{ik\left(jh_x - \frac{-\varphi(\kappa)}{kh_t}\ell h_t\right)}$$

When is the scheme dissipative?

if $|s(k)| < 1$.

What is the phase speed?

$$v_{ph} = -\frac{\varphi(kh_x)}{kh_t}$$

Dispersion?

v_{ph} indep. of k.

if not: scheme dispersive

$$u_t + a u_x = 0$$

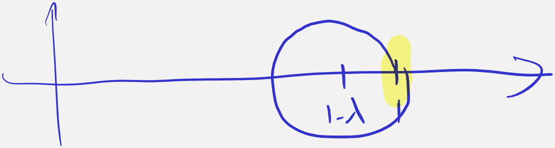
$$u_t = D u_{xx} \qquad (D > 0)$$

$$u_t + a u_x + b u_{xxx} = 0$$

Dispersion/Dissipation Analysis of ETBS

Let $\lambda = ah_t/h_x$. Shown earlier: $s(kh_x) = 1 - \lambda(1 - e^{-ikh_x})$.

$|s(kh_x)| < 1$



Well-resolved waves won't see much dissipation.

Dispersion/Dissipation Analysis of ETBS: Fine Grid

$$e^{-i\omega(\kappa)h_t} = 1 - \lambda(1 - e^{-ikh_x})$$

If kh_x is small, $e^{-ikh} \simeq 1 - ikh_x$.

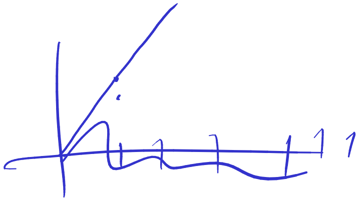
$$s(\kappa) \approx (1-\lambda) + \lambda(1 - i\kappa) = 1 - i\lambda\kappa,$$

If $\omega(\kappa)h_t$ small, approximate $e^{-i\omega(\kappa)h_t} \approx 1 - i\omega(\kappa)h_t$

$$1 - \omega(\kappa)h_t \simeq 1 - i\lambda\kappa \Rightarrow \omega(kh_x)h_t \simeq \lambda h_k = \frac{ah_t}{h_x}h_x = ah_t k$$

$$\omega(kh_x) \simeq ak \quad / \quad v_{ph} \simeq \frac{(-ak)}{kh_t} = -\frac{a}{h_t}$$

Dispersion/Dissipation: Demo



- **Demo:** Experimenting with Dispersion and Dissipation
- **Demo:** Dispersion and Dissipation