

## Today

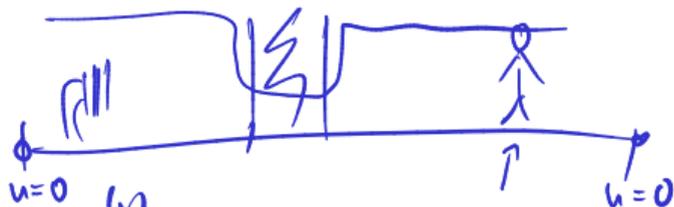
- diss/disp. experiments
- $u_t = -au_x$  ← hyp.  
↳  $u_t = au_{xx}$  ← parabolic
- nonlinear ✓

## Announcements

- HW2 due
- HW3 out later today
- Office Hours All  
Fri @ 4:30 pm

# Heat Equation

Heat equation ( $D > 0$ ):



$$\begin{aligned} \rightarrow u_t &= Du_{xx}, & (x, t) \in \mathbb{R} \times (0, \infty), \\ u(x, 0) &= g(x) & x \in \mathbb{R}. \end{aligned}$$

Fundamental solution ( $g(x) = \delta(x)$ ):

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

$\uparrow$   
 $-(x/2\sqrt{t})^2$

Why is this a weird model?

Infinite speed of prop

# Schemes for the Heat Equation

Cook up some schemes for the heat equation.

Explicit Euler:

$$\frac{u_{k,l+1} - u_{k,l}}{h_t} = \frac{u_{k+1,l} - 2u_{k,l} + u_{k-1,l}}{h_x^2} = 0$$

Implicit Euler:

$$\frac{u_{k,l+1} - u_{k,l}}{h_t} = \frac{u_{k+1,l+1} - 2u_{k,l+1} + u_{k-1,l+1}}{h_x^2} = 0$$

## Von Neumann Analysis of Explicit Euler for Heat (1/2)

Let  $\lambda = Dh_t/h_x^2$ .

$$u_{k,l+1} = u_{k,l} + \lambda(u_{k+1,l} - 2u_{k,l} + u_{k-1,l}).$$

$$\rho_k = I \quad Q_k = \text{bridtrag}(\lambda, 1-2\lambda, \lambda)$$

$$\hat{p}(\varphi) = 1$$

$$\hat{q}(\varphi) = \lambda e^{-i\varphi} + (1-2\lambda) + \lambda e^{i\varphi} = 1-2\lambda + 2\lambda \cos(\varphi)$$

$$|\lambda| \leq 1$$

$$-1 \leq 1 + 2\lambda(\cos(\varphi) - 1) \leq 1$$

$$-2 \leq 2\lambda(\cos(\varphi) - 1) \leq 0$$

## Von Neumann Analysis of Explicit Euler for Heat (2/2)

$$-2 \leq 2\lambda(\cos(\varphi) - 1) \leq 0.$$

$$-2 \leq \cos\varphi - 1 \leq 0$$

$$-2 \leq -4\lambda \Leftrightarrow \frac{1}{2} \geq \frac{Dh_t}{h_x^2} \Leftrightarrow h_t \leq \frac{h_x^2}{2D}$$

$$h_t = O(h_x^2)$$

Comment on the stability region found regarding speeds of propagation.

Explicit scheme propagates at 1 cell/steps.

## Von Neumann Analysis of Implicit Euler for Heat

Let  $\lambda = Dh_t/h_x^2$ .

$$u_{k,l+1} - \lambda(u_{k+1,l+1} - 2u_{k,l+1} + u_{k-1,l+1}) = u_{k,l}$$

$$P_h = \text{tridiag}[-\lambda, 1+2\lambda, -\lambda], \quad Q_h = I$$
$$\hat{p}(\varphi) = 1 + 2\lambda(1 - \cos(\varphi)), \quad \hat{q}(\varphi) = 1$$
$$|s| \leq 1 \Leftrightarrow \left| \frac{1}{\hat{p}(\varphi)} \right| \Leftrightarrow 1 \leq |1 + 2\lambda(1 - \cos\varphi)|$$

infinite  
speed of  
prop.

↳ unconditionally stable

Does the type of system we need to solve for implicit+parabolic correspond to another PDE?

elliptic

# Outline

Introduction

Finite Difference Methods for Time-Dependent Problems

**Finite Volume Methods for Hyperbolic Conservation Laws**

Theory of 1D Scalar Conservation Laws

Numerical Methods for Conservation Laws

Godunov's Method / Reconstruct-Evolve-Average

Limiters

Finite Volume in 2D

Finite Element Methods for Elliptic Problems

Discontinuous Galerkin Methods for Hyperbolic Problems

A Glimpse of Integral Equation Methods for Elliptic Problems

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## Conservation Laws: Recap

$$u_t + f(u)_x = 0, \quad \leftarrow \int_a^b dx$$

where  $u$  is a function of  $x$  and  $t \in \mathbb{R}_0^+$ .

Rewrite in integral form:

$$\frac{d}{dt} \int_a^b u(x,t) dx + f(u(b,t)) - f(u(a,t)) = 0$$

for all  $a, b$

Recall: **Characteristic Curve**: a function  $x(t)$  so that  $u(x(t), t) = u(x_0, 0)$ .

characteristic  
speed:  $f'(u)$

$$\begin{cases} \frac{dx(t)}{dt} = f'(u(x(t), t)), \\ x(0) = x_0. \end{cases}$$



What assumption underlies all this?

differentiability

# Burger's Equation

$$u_t + f(u)_x = 0 \Rightarrow u_t + f'(u)u_x = 0$$

Consider **Burgers' Equation**:



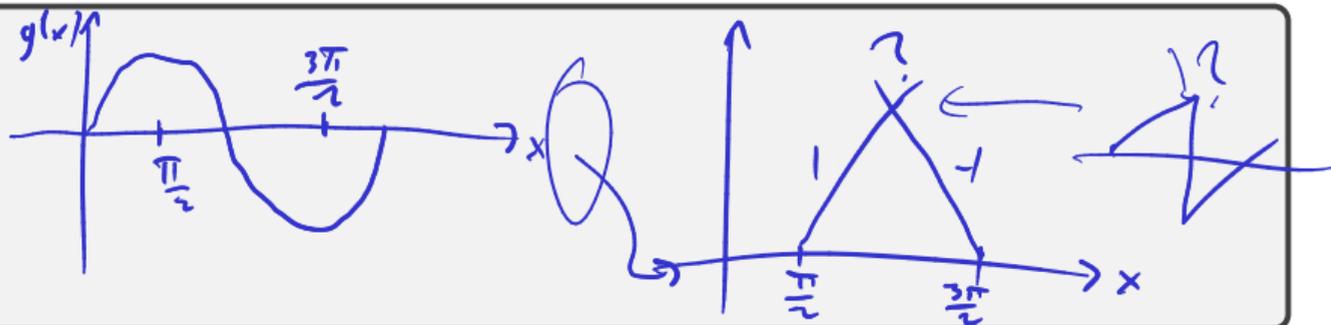
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, \\ u(x, 0) = g(x) = \sin(x). \end{cases}$$

$$u_t + u u_x = 0$$

Interpret Burger's equation.

$$f'(u) = u$$

Consider the characteristics at  $\pi/2$  and  $3\pi/2$ .



## Weak Solutions

$$\frac{d}{dt} \int_a^b u(x, t) dx = f(u(a, t)) - f(u(b, t))$$

Define a weak solution:

- If  $u$  satisfies  $\frac{d}{dt} \int_a^b u(x, t) dx = f(u(a, t)) - f(u(b, t))$  for almost all  $(a, b)$ , then  $u$  is called a weak solution

-  $\varphi \in C_0^1(\mathbb{R}^2)$   $\leftarrow$  "compact support"

$$-\int_0^\infty \int_{-\infty}^\infty u \varphi_t + p(u) \varphi_x dx dt - \int_{-\infty}^\infty g(x) \varphi(x, 0) dx = 0$$

equivalent

# Rankine-Hugoniot Condition (1/2) $\int_{c^-}^{c^+} \rightarrow x$

Consider: Two  $C^1$  segments separated by a curve  $x(t)$  with no regularity.

$$\frac{d}{dt} \left( \underbrace{\int_a^{x(t)} u(x,t) dx}_{G_a(x(t),t)} + \int_{x(t)}^b u(x,t) \right) + f(u(b,t)) - f(u(a,t)) = 0$$

$$\frac{d}{dt} G_a(x(t),t) = \frac{\partial G_a(x(t),t)}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial G_a}{\partial t}$$

$$= u(x(t),t) x'(t) + \int_a^{x(t)} u_t(x,t) dx$$

↑