Today

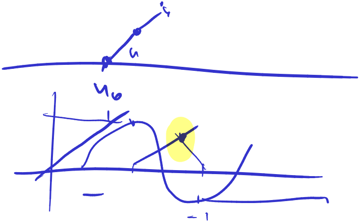
- $u_t + f(u)_x = 0$
  - ↪ characteristic

$u_t + \left(\frac{u^2}{2}\right)_x = 0$

Burgers' → u

characteristic spd. $f'(u)$

↪ straight line



Announcements

- HW 3 out
- AK Office Hours → 4:30pm
- Malachi Office Hours
  → Monday 10am, 0207 Siebel

$u_t = -a u_x$

Weak Solutions

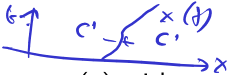
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x = f(u(a,t)) - f(u(b,t))$$ for almost all (a,b)

Define a weak solution:

$\varphi \in C_0^1$

$$-\int_0^\infty \int_{-\infty}^{\infty} u\varphi_t + f(u)\varphi_x \, dx\, dt - \int_{-\infty}^{\infty} u^0(x)\varphi(x,0)\, dx = 0$$

Rankine-Hugoniot Condition (1/2)



Consider: Two C^1 segments separated by a curve $x(t)$ with no regularity.

$$\frac{d}{dt}\left(\underbrace{\int_a^{x(t)} u(x,t)\,dx}_{G_a(x(t),t)} + \int_{x(t)}^{b} u(x,t)\,dx \right) + f(u(b,t)) - f(u(a,t)) = 0$$

$$\frac{d}{dt} G_a(x(t),t) = \frac{\partial G_a(x,t)}{\partial x} \cdot x'(t) + \frac{\partial G}{\partial t}$$

$$= u(x(t),t)\, x'(t) + \int_a^{x(t)} u_t(x,t)\,dx$$

$$= u(x(t),t)\, x'(t) - \int_a^{x(t)} f(u(x,t))_x\,dx$$

$$= u(x(t),t)\, x'(t) - \left(f(u(x(t),t)) - f(u(a,t)) \right)$$

Rankine-Hugoniot Condition (2/2)

$$(d/dt)G_a(x(t),t) = u(x(t),t)x'(t) - (f(u(x(t),t)) - f(u(a,t))).$$

At $(x(t),t)$, doesn't exist. But left and right limits do.

$$\left[\frac{d}{dt}G_a(x(t),t)\right]^- = u^- x'(t) - (f(u^-) - f(u(a,t)))$$

$$\left[\frac{d}{dt}G_b(x(t),t)\right]^+ = -u^+ x'(t) - (f(u(b,t)) - f(u^+))$$

u^- left limit, u^+ right limit

$$u^- x'(t) - f(u^-) - u^+ x'(t) + f(u^+) = 0$$

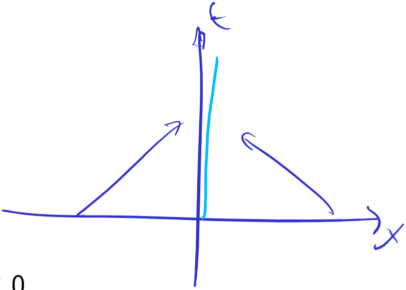
$$x'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{[f(u)]}{[u]}.$$

Rankine-Hugoniot and Weak Solutions

Theorem (Rankine-Hugoniot and Weak Solutions)

If u is piecewise C^1 and is discontinuous only along isoated curves, and if u satisfies the PDE when it is C^1, and the Rankine-Hugoniot condition holds along all discontinuous curves, then u is a weak solution of the conservation law.

Riemann Problems: Example 1



Consider the following Riemann problem:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

$$u(x,0) = \begin{cases} 1 & x < 0, \\ -1 & x >= 0. \end{cases}$$

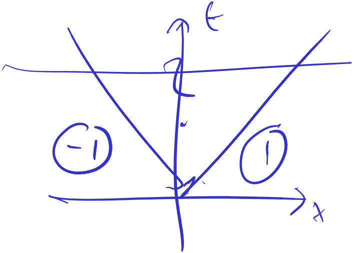
$$x'(t) = \frac{[f(u)]}{[u]} = 0$$

Riemann Problems: Example 2

$f(u) = \frac{u^2}{2}$

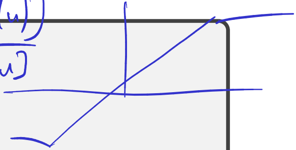
$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

$$u(x,0) = \begin{cases} -1 & x < 0, \\ 1 & x > 0. \end{cases}$$



(IC sign flip compared to previous slide)

Rankine-Hugoniot says $x'(t) = 0 = \frac{[f(u)]}{[u]}$



$$u(x,t) = \begin{cases} -1 & x \le -t \\ x/t & \text{(in btw.)} \\ 1 & x > t \end{cases}$$

↳ rarefaction waves

Ad-Hoc Idea: Ban Bad Shocks

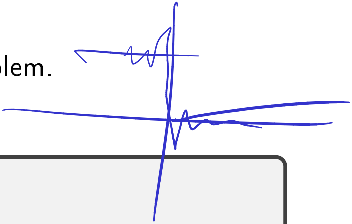
In the shock version of the 'ambiguous' Riemann problem, where do the characteristics go?

out of the shock

Comment on the stability of that situation.

not

Vanishing Viscosity Solutions



Goal: neither uniqueness nor existence poses a problem.

How?

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon$$

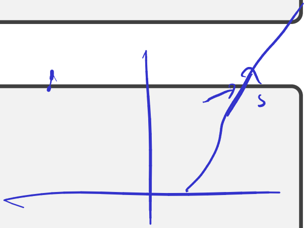
$$\text{"} \lim_{\varepsilon \to 0} u^\varepsilon(x,t) = u(x,t) \text{"}$$

Entropy Conditions: Attempt I

Recall: what is $f'(u)$?

characteristic speed

Devise a way to ban unstable shocks.



$$s = \frac{[f(u)]}{[u]}$$

$$f'(u^-) > s > f'(u^+)$$

entropy condition

If f is convex, $f''(u) \geq 0$, f' nondecreasing: $f'(u^-) > f'(u^+)$

$$u^- \geq u^+$$

Entropy-Flux Pairs

What are features of (physical) entropy?

- constant along particle paths in smooth flow
- entropy increases at shocks

Definition (Entropy/Entropy Flux)

An entropy $\eta(u)$ and an entropy flux $\psi(u)$ are functions so that η is convex and

$$\eta(u)_t + \psi(u)_x = 0$$

for smooth solutions of the conservation law.

$$u_t + f(u)_x = 0$$

Finding Entropy-Flux Pairs

$\eta(u)_t + \psi(u)_x = 0$. Find conditions on η and ψ.

$$\eta'(u) u_t + \psi'(u) u_x = 0$$

$$u_t + f'(u) u_x = 0$$

$$\eta'(u) u_t + \eta'(u) f'(u) u_x = 0$$

$$\psi'(u) = \eta'(u) f'(u)$$

Come up with an entropy-flux pair for Burgers.