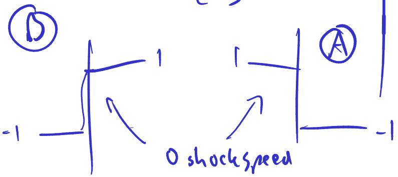
Today

$$\frac{[f(u)]}{[u]} = s$$



Announcements

- HW3 due in a week

$$u_t + a u_x = 0$$

$$f(u) = au$$

$$u_t + f(u)_x = 0$$

$f(u) = \frac{u^2}{2}$ ← Burgers

$f'(u)$

Vanishing Viscosity Solutions

Goal: neither uniqueness nor existence poses a problem.

How?

$$u^\varepsilon_t + f(u)_x = \varepsilon u^\varepsilon_{xx}$$

$$\text{"}\lim_{\varepsilon \to 0}\text{"}\ u^\varepsilon = u$$

viscosity solution

Entropy-Flux Pairs

$u_t + f(u)_x = 0$

What are features of (physical) entropy?

Definition (Entropy/Entropy Flux)

An entropy $\eta(u)$ and an entropy flux $\psi(u)$ are functions so that η is convex and

$$\eta(u)_t + \psi(u)_x = 0$$

for smooth solutions of the conservation law.

Finding Entropy-Flux Pairs

$\eta(u)_t + \psi(u)_x = 0$. Find conditions on η and ψ.

assume $\eta \in C^2$

Assume smooth u.

$$\eta'(u) u_t + \psi'(u) u_x = 0$$

$$u_t + f'(u) u_x = 0 \quad \Big| \cdot \eta'(u)$$

$$\eta'(u) u_t + f'(u) u_x \eta'(u) = 0$$

$$\psi'(u) = \eta'(u) f'(u)$$

Come up with an entropy-flux pair for Burgers.

$f(u) = \frac{u^2}{2}$ $\quad \eta'(u) = e^u$

$\eta(u) = e^u$ $\quad \psi'(u) = u \cdot e^u$

$\eta(u) = u^2$ $\quad \psi'(u) = 2u \cdot u$ $\quad \psi(u) = \frac{2}{3} u^3$

Back to Vanishing Viscosity (1/2)

$$u_t + f(u)_x = \varepsilon u_{xx}$$

What's the evolution equation for the entropy?

$$\eta'(u) \cdot u_t + \eta'(u) \cdot f'(u)\, u_x = \varepsilon \eta'(u)\, u_{xx}$$

$$\eta(u)_t + \psi(u)_x = \varepsilon\left(\eta'(u) u_x\right)_x - \varepsilon \underbrace{\eta''(u)}_{\geq 0} u_x^2$$

Back to Vanishing Viscosity (2/2)

$$\eta(u)_t + \psi^{\cancel{\prime}}(u)_x = \varepsilon(\eta'(u)u_x)_x - \varepsilon\eta''(u)u_x^2.$$

Integrate this over $[x_1, x_2] \times [t_1, t_2]$.

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \eta(u)_t + \psi(u)_x \, dx \, dt$$

$$= \varepsilon\int_{t_1}^{t_2}\left[\eta'(u(x_2,t))u_x(x_2,t) - \eta'(u(x_1,t))u_x(x_1,t)\right]dt$$

$\to 0$ as $\varepsilon \to 0$

$$- \varepsilon\int_{t_1}^{t_2}\int_{x_1}^{x_2} \underbrace{\eta''(u)u_x^2}_{\geq 0} \, dx \, dt$$

does not go to 0 as $\varepsilon \to 0$

Weakly:

$$\eta(u)_t + \psi(u)_x \leq 0$$

Entropy solution

Definition (Entropy solution)

The function $u(x,t)$ is the entropy solution of the conservation law if for all convex entropy functions and corresponding entropy fluxes, the inequality

$$\eta(u)_t + \psi(u)_x \leq 0$$

is satisfied in the weak sense.

showed: viscosity solution $\Rightarrow$ entropy solution

Conservation of Entropy?

What can you say about conservation of entropy in time?

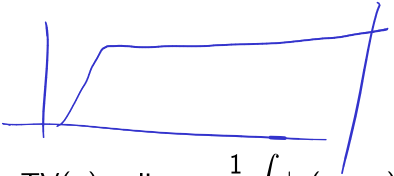
$$0 \geq \int_{t_1}^{t}\int_{x_1}^{x_2} \eta(u)_t + \psi(u)_x \, dx\, dt$$

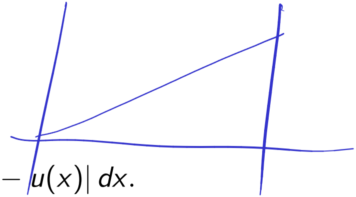
$$= \left[\int_{x_1}^{x_2} \eta(u(x,t))\, dx\right]_{t_1}^{t_2} - \left[\int_{t_1}^{t_2} \psi(u(x,t))\, dt\right]_{x_1}^{x_2}$$

$$\int_{x_1}^{x_2} \eta(u(x,t_2))\, dx \leq \int \int_{x_1}^{x_2} \eta(u(x,t_1)) - \sim$$

$$u_t + f(u)_x = 0$$

Total Variation





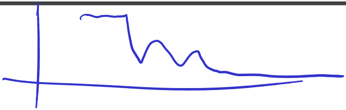
$$\mathrm{TV}(u) = \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int |u(x+\varepsilon) - u(x)| \, dx.$$

Simpler form if u is differentiable?

$$\mathrm{TV}(u) = \int |u'| \, dx$$

Hiking analog?

Elevation change



Total Variation and Conservation Laws

Theorem (Total Variation is Bounded)

Let u be a solution to a conservation law with $f''(u) \geqslant 0$. Then:

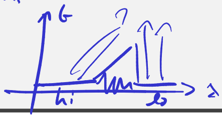
$$\mathrm{TV}(u(t+\Delta t,\cdot)) \leq \mathrm{TV}(u(t,\cdot)) \quad \textit{for } \Delta t \geqslant 0.$$

TVB

TVD

– For smooth solution, (using characteristics) TV is conserved.

– For non smooth: TV decreases



Theorem (L^1 contraction)

Let u, v be viscosity solutions of the conservation law. Then

$$\|u(t+\Delta,\cdot) - v(t+\Delta t,\cdot)\|_{L^1(\mathbb{R})} \leq \|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})} \quad \textit{for } \Delta t \geqslant 0.$$

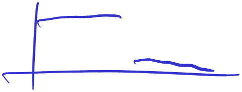
Outline

Finite Difference for Conservation Laws? (1/2)

$x(t) = ut + u(x,0)$

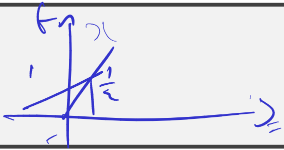
$f'(u) = u$

$s = \frac{[f(u)]}{[u]} \approx f'(u)$



$$\begin{cases} u_t + \left(\frac{u}{2}\right)^2_x = 0 \\ u(x,0) = \begin{cases} 1 & x < 0, \\ 0 & x \geqslant 0. \end{cases} \end{cases}$$

Entropy Solution?



Rewrite the PDE to 'match' the form of advection $u_t + au_x = 0$:

$u_t + uu_x = 0$

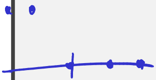
Equivalent?

Finite Difference for Conservation Laws? (2/2)

Recall the *upwind scheme* for $u_t + au_x = 0$: FTBS, $a > 0$

$$u_{j,\ell+1} = u_{j,\ell} - a\frac{\Delta t}{\Delta x}\left(u_{j,\ell} - u_{j-1,\ell}\right)$$

Write the upwind FD scheme for $u_t + uu_x = 0$:



$$u_{j,\ell+1} = u_{j,\ell} - u_{j,\ell}\frac{\Delta t}{\Delta x}\left(u_{j,\ell} - u_{j-1,\ell}\right)$$

For $j \neq 0$: $u_{j,\ell} - u_{j-1,\ell} = 0$

For $j = 0$: $u_{j,\ell} = 0$

Schemes in Conservation Form

Definition (Conservative Scheme)

A conservation law scheme is called conservative iff it can be written as

where $f^* \ldots$

Theorem (Lax-Wendroff)

If the solution $\{u_{j,\ell}\}$ to a conservative scheme converges (as $\Delta t, \Delta x \to 0$) boundedly almost everywhere to a function $u(x,t)$, then u is a weak solution of the conservation law.