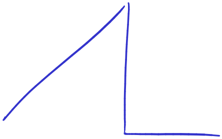
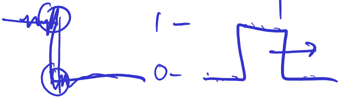
Today

Announcements

- HW3 → next Wed

TVD





$$u_\ell(\varphi) = g_+(\varphi)^\ell a(\varphi) + g_-(\varphi) b(\varphi)$$

$$\sup_{\{x_j\}} \{ |u(x_{j+1}) - u(x_j)|$$

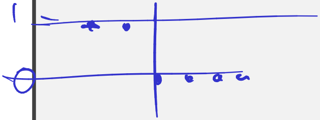
Finite Difference for Conservation Laws? (2/2)

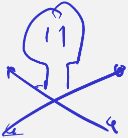
Recall the *upwind scheme* for $u_t + au_x = 0$:

$$u_{j,\ell+1} = u_{j,\ell} - a\frac{h_t}{h_x}\left(u_{j,\ell} - u_{j-1,\ell}\right)$$

Write the upwind FD scheme for $u_t + uu_x = 0$:

$$u_{j,\ell+1} = u_{j,\ell} - u_{j,\ell}\frac{h_t}{h_x}\left(u_{j,\ell} - u_{j-1,\ell}\right)$$





Schemes in Conservation Form $u_t + f(u)_x$ $\frac{d}{dt}\int_a^b u - f(u(a,0)) + f(u(b,0))$

Definition (Conservative Scheme)

A conservation law scheme is called conservative iff it can be written as

$$u_{j,\ell+1} = u_{j,\ell} - \frac{h_t}{h_x}\left[f^*_{j+\frac{1}{2}}(\vec{u}_\ell) - f^*_{j-1/2}(\vec{u}_\ell)\right]$$

where f^*...

- Lipschitz continuous ←
- $f^*(u, \ldots, u) = f(u)$ ←

Theorem (Lax-Wendroff)

If the solution $\{u_{j,\ell}\}$ to a conservative scheme converges (as $\Delta t, \Delta x \to 0$) boundedly almost everywhere to a function $u(x,t)$, then u is a weak solution of the conservation law.

Lax-Wendroff Theorem: Proof

Summation by parts: With $\Delta^+ a_k = a_{k+1} - a_k$ and $\Delta^- a_k = a_k - a_{k-1}$:

$$\sum_{k=1}^{N} a_k(\Delta^- \varphi_k) + \sum_{k=1}^{N} \varphi_k(\Delta^+ a_k) = -a_1\varphi_0 + \varphi_N a_{N+1}.$$

$f^*_{1+1/2}$ $f^*_{2+1/2}$ $f^*_{3+1/2}$

Let $\varphi_{j,\ell} = \varphi(x_j, t_\ell)$ $\varphi \in C_0^1$

$$0 = \sum_{\ell=1}^{\infty} \sum_j \left(\frac{\Delta_2^+ u_{j,\ell}}{h_t} + \frac{\Delta^+ f^*_{j-1/2}}{h_x} \right) \varphi_{j,\ell} h_t h_x$$

$$= -\sum_{\ell=1}^{\infty} \sum_j \left(\frac{\Delta_2^- \varphi_{j,\ell}}{h_t} u_{j,\ell} + \frac{\Delta_1^- \varphi_{j,\ell}}{h_x} f^*_{j-1/2} \right) h_x h_t - \sum_j u_{j,1} \varphi_{j,0} h_x$$

$$\xrightarrow{DCT} -\int_0^\infty \int_{-\infty}^\infty \varphi_t u + \varphi_x f(u)\, dx\, dt - \int_{-\infty}^\infty u(x,0)\varphi(x,0)\, dx$$

Riemann Problem

Consider the Riemann problem:

$$\begin{cases} u_t + f(u)_x = 0, \\ u(x,0) = \begin{cases} u_l & x < 0, \\ u_r & x \geqslant 0 \end{cases} \end{cases}$$

Exact solution in the Burgers case?