Today

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = \begin{cases} u_\ell & x<0 \\ u_r & x \geq 0 \end{cases} \end{cases}$$

- FV methods
  - ↳ conservative schemes
- Riemann solvers
- high order

$$s = \frac{[f(u)]}{[u]}$$

Announcements

- Quiz available
- HW3 due today
- Will post project tomorrow, due April 8
- Will also post HW4 this week (end) due , 1
- Virus plans

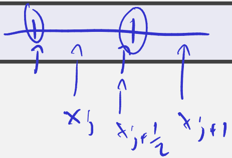
Schemes in Conservation Form $u_t + f(u)_x = 0$

Definition (Conservative Scheme)

A conservation law scheme is called conservative iff it can be written as

$$u_{j,\ell+1} = u_{j,\ell} - \frac{\Delta t}{\Delta x}\left[f^*_{j+1/2}(\bar{u}_\ell) - f^*_{j-\frac{1}{2}} \right]$$

where f^*...



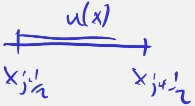
Theorem (Lax-Wendroff)

If the solution $\{u_{j,\ell}\}$ to a conservative scheme converges (as $\Delta t, \Delta x \to 0$) boundedly almost everywhere to a function $u(x,t)$, then u is a weak solution of the conservation law.

Finite Volume Schemes

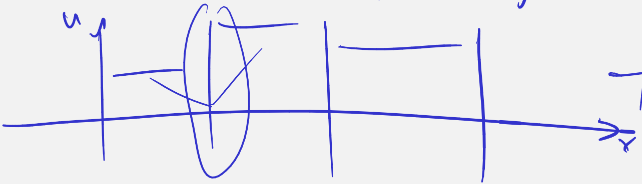
DoFs: the numbers that represent the solution

Finite volume: Idea?



- $\bar{u}_j = \dfrac{1}{h_x}\displaystyle\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x)\,dx$

cell averages



prevents shocks from hitting adjacent interfaces $\to \max|f'(u)| \cdot h_t \le h_x$

Developing Finite Volume

$$u(t_{\ell+1}) - u(t_\ell) \leftarrow \int_{t_\ell}^{t^{\ell+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} (u_t + f(u)_x)\mathrm{d}x\mathrm{d}t = 0$$

$$\frac{1}{h_x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^{\ell+1} dx - \frac{1}{h_x}\int_{x_{j-1/2}}^{x_{j+\frac{1}{2}}} u^{\ell} dx$$

$$+ \frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j+\frac{1}{2}})\, dt - \frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j-\frac{1}{2}})\, dt = 0$$

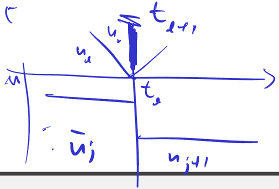
$$\bar{u}_{j,\ell+1} - \bar{u}_{j,\ell}$$

$$+ \frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j+\frac{1}{2}})\, dt - \frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j-\frac{1}{2}})\, dt = 0$$

Flux Integrals?

$u_t + f(u)_x = 0$

$$\frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j+1/2})\,\mathrm{d}\cancel{x}t?$$



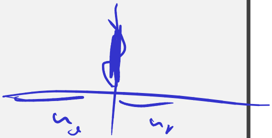
Consider the substitution

$\bar{x} = a x \qquad \bar{t} = a t$

Does this change the IC? no
PDE? no

} Riemann problem stays unchanged

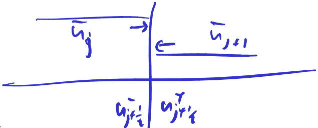
$\Rightarrow u(x_{j+\frac{1}{2}}, t) = c \quad \forall t \in (t_\ell, t_{\ell+1})$



$$\frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j+\frac{1}{2}})\,dt = \frac{h_t}{h_x} f(u_{j+\frac{1}{2}})$$

The Godunov Scheme

Altogether:



$$u_t + f(u)_x = 0 \qquad \bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} - \frac{h_t}{h_x}\left(f(u_{j+1/2,\ell}) - f(u_{j-1/2,\ell})\right).$$

Overall algorithm?

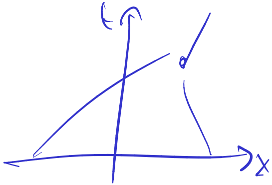
- Reconstruct $u^{\pm}_{j\pm\frac12}$, e.g. $u^-_{j+\frac12} = \bar{u}_j$
- Evolve the Riemann problem $f^*(u^-_{j+\frac12}, u^+_{j+\frac12})$ — numerical flux
- Averages the Riemann solutions

Heuristic time step restriction?

$$h_t \le h_x / \max_j |f'(u_j)|$$

Riemann Problem

Consider the Riemann problem:



$$\begin{cases} u_t + f(u)_x = 0, \\ u(x,0) = \begin{cases} u_l & x < 0, \\ u_r & x \geqslant 0 \end{cases} \end{cases}$$

Exact solution in the Burgers case? $\left(f(u) = \frac{u^2}{2} \right)$

$$u(x,t) = \begin{cases} \begin{cases} u_l & x < st \\ u_r & x \geq st \end{cases} & u_l \geq u_r \quad \leftarrow \text{shock} \\ \begin{cases} u_l & x < u_l t \\ x/t & \text{otherwise} \\ u_r & x \geq u_r t \end{cases} & \text{otherwise} \quad \leftarrow \text{rarefaction} \end{cases}$$

$$s = \frac{f(u_r) - f(u_l)}{u_r - u_l}$$

Riemann Problems in General

Analytically solvable?

Can information only travel at finite speed in conservation laws? Why?

Rieamnn Solver for a General Conservation Law

To complete the scheme: Need $f^*(u^-, u^+)$. For Burgers: already known.
For a general (convex/concave-f) conservation law?

$f''(u) \geq 0$

$$s = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

$$f^*(u^-, u^+) = \begin{cases} f(u^-) & \text{if shock, if } s > 0 \\ f(u^+) & \text{if shock, if } s \leq 0 \\ f(u^-) & \text{if rarefaction, } f'(u^-) \geq 0 \\ f(u^+) & \text{if rarefaction, } f'(u^+) \leq 0 \\ f(u_s) & \text{if rarefaction, } f'(u^-) \leq 0 \leq f'(u^+) \end{cases}$$

Find u_s ("stagnation state") so that $f'(u_s) = 0$

Equivalent to

$$f^*(u^-, u^+) = \begin{cases} \max_{u^+ \leq u \leq u^-} f(u) & \text{if } u^- > u^+, \\ \min_{u^- \leq u \leq u^+} f(u) & \text{if } u^- < u^+. \end{cases}$$

More Riemann Solvers

Downside of Godunov Riemann solver?

argh, too complicated

Back to Advection

Consider only $f(u) = au$ for now. Riemann solver inspiration from FD?

$a \geq 0$, ETBS

$$0 = \frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + a\frac{u_{j,\ell} - u_{j-1,\ell}}{h_x}$$

$$= \frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f(u_{j,\ell}) - f(u_{j-1,\ell})}{h_x}$$

$$= \frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f^*(\underbrace{u_{j,\ell}}_{u^-}, \underbrace{u_{j+1,\ell}}_{u^+}) - f^*(\underbrace{u_{j-1,\ell}}_{u^-}, \underbrace{u_{j,\ell}}_{u^+})}{h_x}$$

$$f^*(u^-, u^+) = f(u^-)$$

implies ETBS can be interpreted as a FV scheme

For general a:

$$f^*(u^-, u^+) = \begin{cases} f(u^-) & a \geq 0 \\ f(u^+) & a < 0 \end{cases} = \underbrace{\frac{au^- + au^+}{2} - \frac{|a|}{2}(u^+ - u^-)}$$

upwind flux = Lax-Friedrichs flux