Today

$$u_t + f(u)_x = 0$$

- Godunov scheme
- upwind

Announcements

HW3 due today

Project posted tomorrow (Piazza announcement)

HW4 after break

Livestream

↳ instant msg

Office Hours

↳ URL on web page

More Riemann Solvers

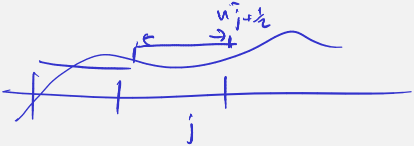
Downside of Godunov Riemann solver?

too much work

Back to Advection

Consider only $f(u) = au$ for now. Riemann solver inspiration from FD?

$$\bar u_{j,\ell+1} = \bar u_{j,\ell} - \frac{h_t}{h_x}\left(f^*(\overset{u^-}{\bar u_j}, \overset{u^+}{\bar u_{j+1}}) - f^*(\bar u_{j-1}, \bar u_j) \right)$$



$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{|a|}{2}(u^+ - u^-) = \begin{cases} au^- & a > 0 \\ au^+ & a \le 0 \end{cases}$$

Side Note: First Order Upwind, Rewritten

$$\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f^*(u_{j,\ell}, u_{j+1,\ell}) - f^*(u_{j,\ell-1}, u_{j,\ell})}{h_x}$$

with

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{|a|}{2}(u^+ - u^-). \quad \leftarrow f(u) = au$$

$$\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + a\frac{u_{j+1} - u_{j-1}}{2h_x} = \frac{|a| h_x}{2} \frac{u_{j+1} - 2u_j + u_{j-1}}{h_x^2}$$

ETCS

Lax-Friedrichs Fluxes

Generalize linear upwind flux for a nonlinear conservation law:

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{|a|}{2}(u^+ - u^-).$$

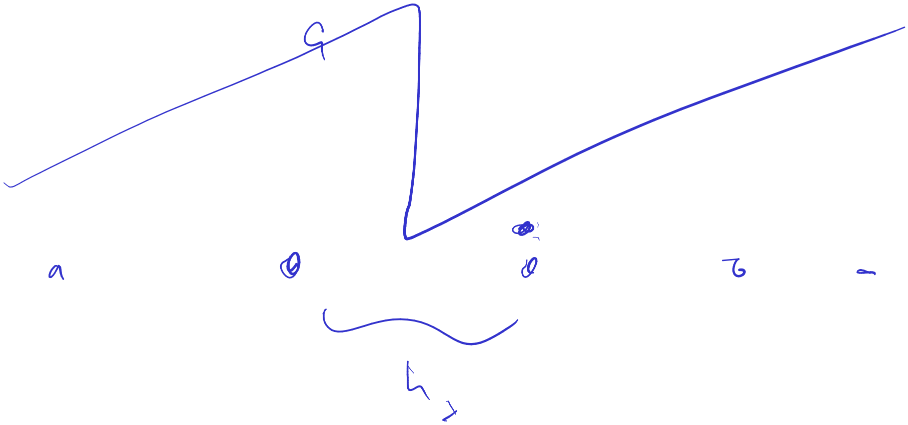
$$f^*(u^-, u^+) = \frac{f(u^-) + f(u^+)}{2} - \frac{\alpha}{2}(u^+ - u^-)$$

$\alpha \rightsquigarrow |f'|$-ish ~~$\alpha = |f((u^- + u^+)/2)|$~~ bad idea

Rusanov / local LF: $\alpha_1 = \max(|f'(u^-)|, |f'(u^+)|)$

global LF: $\alpha_2 = \max_j(|f'(u_j)|)$

Demo: Finite Volume Burgers (Part I)



Outline

Improving Accuracy

Consider our existing discrete FV formulation:

$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} - \frac{h_t}{h_x}(f(u_{j+1/2,\ell}) - f(u_{j-1/2,\ell})).$$

What obstacles exist to increasing the order of accuracy?

- Temporal accuracy
- Spatial accuracy
- Shocks

What order of accuracy can we expect?

- @ shock: L^∞ nothing, L^2 maybe first
- otherwise as much as we want

Improving the Order of Accuracy

Improve temporal accuracy.

$$\frac{d\bar{u}_j(t)}{dt} + \frac{f^*(u^-_{j+1/2}, u^+_{j+1/2}) - f^*(u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}})}{h_x} = 0$$

What's the obstacle to higher spatial accuracy?

$$u_{j+\frac{1}{2}} = \bar{u}_j \,?$$

How can we improve the accuracy of that approximation?

involve more cells in reconstruction

Increasing Spatial Accuracy

Temporary Assumptions:

- $f'(u) \geqslant 0$
- $\hat{f}_{j+1/2} = f(\bar{u}_j)$ (e.g. Godunov in this situation)

$f^*(u^-, u^+) = u^-$

Reconstruct $u_{j+1/2}$ using the information $\{\bar{u}_j, \bar{u}_{j+1}\}$. Accuracy? Names?

$\bar{u}_{j-1}$ $\bar{u}_j$ $\bar{u}_{j+1}$ upwind central

$$u^{(1)}_{j+\frac{1}{2}} = \frac{1}{2}\left(\bar{u}_j + \bar{u}_{j+1}\right) \quad \text{(central)}$$

$$u^{(2)}_{j+\frac{1}{2}} = \frac{3}{2}\bar{u}_j - \frac{1}{2}\bar{u}_{j-1} \quad \text{(upwind)}$$

Compute fluxes, use increments over cell average:

$$f^{*(1)}_{j+\frac{1}{2}} = f\Big(\bar{u}_j + \frac{1}{2}\underbrace{\left(\bar{u}_{j+1} - \bar{u}_j\right)}_{\text{slope}}\Big) \Big/ \; f^{*(2)}_{j+\frac{1}{2}} = f\Big(\bar{u}_j + \frac{1}{2}\left(\bar{u}_j - \bar{u}_{j-1}\right)\Big)$$

Lax-Wendroff

For $u_t + au_x$, from finite difference:

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{a^2}{2} \cdot \frac{\Delta t}{\Delta x}(u^+ - u^-).$$

Taylor in time: $u_{\ell+1} = u_\ell + \partial_t u_\ell \cdot h_t + \partial_t^2 u_\ell \cdot h_t/2 + O(h_t^3)$.

$$\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f(u_{j+1,\ell}) - f(u_{j-1,\ell})}{2h_x}$$
$$= \frac{h_t}{2h_x}\left[f'(u_{j+1/2,\ell})\frac{f(u_{j+1,\ell}) - f(u_{j,\ell})}{h_x} - f'(u_{j-1/2,\ell})\frac{f(u_{j,\ell}) - f(u_{j-1,\ell})}{h_x}\right]$$

As a Riemann solver:

$$f^*(u^-, u^+) = \frac{f(u^-) + f(u^+)}{2} - \frac{h_t}{h_x}[f'(u^\circ)(f(u^+) - f(u^-))].$$