Today

- high order
- monotone schemes
- TVD schemes
- proving TVD
- "minmod"
- time discr.

Announcements

- Project 1 posted
- HW4 soon
- Office hours online

Improving Accuracy

Consider our existing discrete FV formulation:

$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} - \frac{h_t}{h_x}\left(f(u_{j+1/2,\ell}) - f(u_{j-1/2,\ell})\right).$$

What obstacles exist to increasing the order of accuracy?

$$\frac{d\bar{u}_j}{dt} = \frac{f^*_{j+\frac{1}{2}} - f^*_{j-\frac{1}{2}}}{h_x}$$

- reconstruction
- nonsmoothness

What order of accuracy can we expect?

- away from shock ('smooth'): as high as we like
- at a shock: first order in L^2

Improving the Order of Accuracy

Improve temporal accuracy.

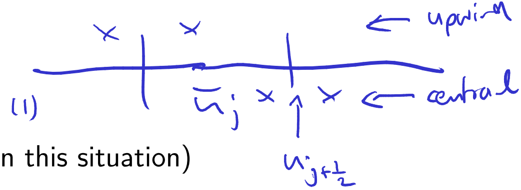
MOL

What's the obstacle to higher spatial accuracy?

reconstruction

How can we improve the accuracy of that approximation?

Increasing Spatial Accuracy



Temporary Assumptions:

- $f'(u) \geqslant 0$
- $\hat{f}_{j+1/2} = f(\bar{u}_j)$ (e.g. Godunov in this situation)

Reconstruct $u_{j+1/2}$ using $\{\bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}\}$. Accuracy? Names?

$$u^{(1)}_{j+\frac{1}{2}} = \frac{1}{2}\left(\bar{u}_j + \bar{u}_{j+1}\right)$$

$$u^{(2)}_{j+\frac{1}{2}} = \frac{3}{2}\bar{u}_j - \frac{1}{2}\bar{u}_{j-1}$$

Compute fluxes, use increments over cell average:

$$\hat{f}^{(1)}_{j+\frac{1}{2}} = f\Big(\bar{u}_j + \frac{1}{2}\overbrace{\left(\bar{u}_{j+1} - \bar{u}_j\right)}^{\tilde{u}^{(1)}_j}\Big)$$

$$\rightarrow \hat{f}^{(2)}_{j+\frac{1}{2}} = f\Big(\bar{u}_j + \frac{1}{2}\overbrace{\left(\bar{u}_j - \bar{u}_{j-1}\right)}^{\tilde{u}^{(2)}_j}\Big)$$

Lax-Wendroff

$u_t + f(u)_x = 0$

For $u_t + au_x$, from finite difference:

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{a^2}{2} \cdot \frac{\Delta t}{\Delta x}(u^+ - u^-).$$

Taylor in time: $u_{\ell+1} = u_\ell + \partial_t u_\ell \cdot h_t + \partial_t^2 u_\ell \cdot h_t/2 + O(h_t^3)$.

$u_t = -f(u)_x$

$u_{tt} = -f(u)_{xt} = -(f(u)_t)_x = -(f'(u)\,u_t)_x = +(f'(u)\,f(u)_x)_x$

$$\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f(u_{j+1,\ell}) - f(u_{j-1,\ell})}{2h_x}$$

$$= \frac{h_t}{2h_x}\left[f'(u_{j+1/2,\ell})\frac{f(u_{j+1,\ell}) - f(u_{j,\ell})}{h_x} - f'(u_{j-1/2,\ell})\frac{f(u_{j,\ell}) - f(u_{j-1,\ell})}{h_x}\right]$$

As a Riemann solver:

$$f^*(u^-, u^+) = \frac{f(u^-) + f(u^+)}{2} - \frac{h_t}{h_x}[f'(u^\circ)(f(u^+) - f(u^-))].$$

Monotone Schemes

Definition (Monotone Scheme)

A scheme

$$\begin{aligned} u_{j,\ell+1} &= u_{j,\ell} - \lambda(\hat{f}(u_{j-p}, \ldots, u_{j+q}) - \hat{f}(u_{j-p-1}, \ldots, u_{j+q-1})) \\ &=: G(u_{j-p-1}, \ldots, u_{j+q}) \end{aligned}$$

is called a montone scheme if G is a monotonically nondecreasing function $G(\uparrow, \uparrow, \ldots, \uparrow)$ of each argument.

Monotonicity for Three-Point Schemes

Three-Point Scheme:

$$G(u_{j-1}, u_j, u_{j+1}) = u_j - \lambda[\hat{f}(u_j, u_{j+1}) - \hat{f}(u_{j-1}, u_j)].$$

$\hat{G}(\uparrow, \uparrow, \uparrow)$?

When is this monotone?

If $\hat{f}(\uparrow, \downarrow)$

$$\frac{\partial G}{\partial u_j} = 1 - \lambda(\hat{f}_1 - \hat{f}_2) \geq 0$$

part. derivative wrt to arg 1.

If $\underbrace{\lambda(\hat{f}_1 - \hat{f}_2) \leq 1}$, then we have monotonicity.

$\hookrightarrow$ CFL restriction

$\lambda = \frac{\Delta t}{\Delta x}$

Lax-Friedrichs is Monotone

$$f^*(u^-, u^+) = \frac{f(u^-) + f(u^+)}{2} - \frac{\alpha}{2}(u^+ - u^-).$$

Show: This is monotone. ✓

$$f_1^* = \frac{1}{2}\left(f'(u_j) + \alpha\right) \geq 0$$

$$f_2^* = \frac{1}{2}\left(f'(u_{j+1}) - \alpha\right) \leq 0$$

$$\alpha = \max_u |f'(u)|$$

Monotone Schemes: Properties

Theorem (Good properties of monotone schemes)

- *Local maximum principle:*

$$\min_{i \in \text{stencil around } j} u_i \leq G(u)_j \leq \max_{i \in \text{stencil around } j} u_i.$$

- *L^1-contraction:*

$$\|G(u) - G(v)\|_{L^1} \leq \|u - v\|_{L^1}.$$

- *TVD:*

$$TV(G(u)) \leq TV(u).$$

- *Solutions to monotone schemes satisfy all entropy conditions.*

Godunov's Theorem

Theorem (Godunov)

Monotone schemes are at most first-order accurate.

What now?

Just ask for TVD ?

Linear Schemes

Definition (Linear Schemes)

A scheme is called a linear scheme if it is linear when applied to a linear PDE:

$$u_t + au_x = 0,$$

where a is a constant.

Write the general case of a linear scheme for $u_t + u_x = 0$:

$$u_j^{\ell+1} = \sum_{k=-K}^{K} c_n(\lambda)\, u_{j-k} \qquad \lambda = \frac{\Delta t}{\Delta x}$$

A linear scheme is monotone if all coeffs $c_k(\lambda) \geq 0$.

Also called positive schemes

Linear + TVD = ?

Theorem (TVD for linear Schemes)

For linear schemes, TVD $\Rightarrow$ monotone.

What does that mean?

Linear TVD schemes are at most first order

Now what?

Nonlinear TVD scheme could be higher order.

Harten's Lemma

Theorem (Harten's Lemma)

If a scheme can be written as $\lambda = \frac{\Delta t}{\Delta x}$

$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} + \lambda\left(C_{j+1/2}\Delta_+\bar{u}_j - D_{j-1/2}\Delta_-\bar{u}_j\right)$$

with $C_{j+1/2} \geqslant 0$, $D_{j+1/2} \geqslant 0$, $1 - \lambda(C_{j+1/2} + D_{j+1/2}) \geqslant 0$ *and* $\lambda = \Delta t/\Delta x$, *then it is TVD.*

As a matter of notation, we have

$$\begin{aligned} \Delta_+ u_j &= u_{j+1} - u_j, \\ \Delta_- u_j &= u_j - u_{j-1}. \end{aligned}$$

We have omitted the time subscript for the time level ℓ.

Harten's Lemma: Proof

$\sum |\bar{u}_{j+1} - \bar{u}_j| = \sum |\Delta_+ \bar{u}_j|$

$$\Delta_+ \bar{u}_{j,\ell+1} = \Delta_+ \bar{u}_j + \lambda \Big(C_{j+\frac{3}{2}} \Delta_+ \bar{u}_{j+1} - D_{j+\frac{1}{2}} \Delta_+ \bar{u}_j - C_{j+\frac{1}{2}} \Delta_+ \bar{u}_j + D_{j-\frac{1}{2}} \Delta_- \bar{u}_j \Big)$$

$$|\Delta_+ \bar{u}_{j,\ell+1}| \leq \Big(1 - \lambda\big(C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}}\big)\Big) |\Delta_+ \bar{u}_j| + \lambda C_{j+\frac{3}{2}} |\Delta_+ \bar{u}_j| + \lambda D_{j-\frac{1}{2}} |\Delta_- \bar{u}_j|$$

$$\sum_j |\Delta_+ \bar{u}_{j,\ell+1}| \leq \sum \big(1 - \lambda(C + D)\big) |\Delta_+| + \lambda \underline{\sum C |\Delta_+|} + \lambda \underline{\sum D |\Delta_-|}$$

$$\leq \sum_j |\Delta_+ \bar{u}_{j,\ell}|$$

Minmod Scheme

$f'(u) \geq 0$

$$\hat{f}^{(1)}_{j+1/2} = f\left(\bar{u}_j + \underbrace{\frac{1}{2}(\bar{u}_{j+1} - \bar{u}_j)}_{\tilde{u}_j^{(1)}}\right), \qquad \hat{f}^{(2)}_{j+1/2} = f\left(\bar{u}_j + \underbrace{\frac{1}{2}(\bar{u}_j - \bar{u}_{j-1})}_{\tilde{u}_j^{(2)}}\right).$$

Design a 'safe' thing to use for $\tilde{u}$:

$$\operatorname{minmod}(a,b) = \begin{cases} a & |a| < |b|,\ ab > 0 \\ b & |b| \leq |a|,\ ab > 0 \\ 0 & ab \leq 0 \end{cases}$$

$$\tilde{u}_j = \operatorname{minmod}\left(\tilde{u}_j^{(1)}, u_j^{(2)}\right)$$

$$f^{(3)}_{j+\frac{1}{2}} = f\left(\bar{u}_j + \tilde{u}_j\right)$$