Today

- TVD
  - ↳ Minmod
  - ↳ SSP
  - ↳ limitations / extensions
- FEM

Announcements

- Feedback form
- Lecture format?

Harten's Lemma

Theorem (Harten's Lemma)

If a scheme can be written as

$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} + \lambda(C_{j+1/2}\Delta_+\bar{u}_j - D_{j-1/2}\Delta_-\bar{u}_j)$$

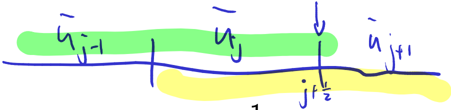
with $C_{j+1/2} \geq 0$, $D_{j+1/2} \geq 0$, $1 - \lambda(C_{j+1/2} + D_{j+1/2}) \geq 0$ *and* $\lambda = \Delta t/\Delta x$, *then it is TVD.*

As a matter of notation, we have

$$\begin{aligned}\Delta_+ u_j &= u_{j+1} - u_j, \\ \Delta_- u_j &= u_j - u_{j-1}.\end{aligned}$$

We have omitted the time subscript for the time level ℓ.

Minmod Scheme



Still assume $f'(u) \geq 0$.

$$f^{*,(1)}_{j+1/2} = f\Big(\bar{u}_j + \underbrace{\frac{1}{2}(\bar{u}_{j+1} - \bar{u}_j)}_{\tilde{u}_j^{(1)}}\Big), \qquad f^{*,(2)}_{j+1/2} = f\Big(\bar{u}_j + \underbrace{\frac{1}{2}(\bar{u}_j - \bar{u}_{j-1})}_{\tilde{u}_j^{(2)}}\Big).$$

Design a 'safe' thing to use for $\tilde{u}$:

$$\operatorname{minmod}(a,b) = \begin{cases} a & |a| \leq |b| \\ b & |b| < |a| \end{cases} \Bigg\} \; ab \geq 0, \qquad 0 \quad ab < 0$$

$$\tilde{u}_j^{(3)} = \operatorname{minmod}\left(\tilde{u}_j^{(1)}, \tilde{u}_j^{(2)}\right)$$

$$f^{*,(3)}_{j+\frac{1}{2}} = f\left(\bar{u}_j + \tilde{u}_j^{3}\right)$$

Minmod is TVD

$$\bar{u}_{j,\ell+1} = \bar{u}_j + \lambda \left(C \Delta_+ - D \Delta_- \right)$$

Show that Minmod is TVD:

$$\bar{u}_{j,\ell+1} = \bar{u}_j - \lambda \left(f(\bar{u}_j + \hat{u}_j) - f(\bar{u}_{j-1} + \hat{u}_{j-1}) \right)$$

$$= \bar{u}_j - \lambda \left(-D_{j-\frac{1}{2}} \Delta_- \bar{u}_j \right)$$

$$D_{j-\frac{1}{2}} = \frac{f(\bar{u}_j + \hat{u}_j) - f(\bar{u}_{j-1} + \hat{u}_{j-1})}{\bar{u}_j - \bar{u}_{j-1}} = f'(\xi) \frac{\bar{u}_j - \bar{u}_{j-1} + \hat{u}_j - \hat{u}_{j-1}}{\bar{u}_j - \bar{u}_{j-1}}$$

$$= \underbrace{f'(\xi)}_{\geq 0} \left[1 + \underbrace{\frac{\hat{u}_j}{\bar{u}_j - \bar{u}_{j-1}}}_{0 \leq \cdot \leq \frac{1}{2}} - \underbrace{\frac{\hat{u}_{j-1}}{\bar{u}_j - \bar{u}_{j-1}}}_{0 \leq \cdot \leq \frac{1}{2}} \right] \geq 0$$

Minmod: CFL restriction?

Derive a time step restriction for Minmod.

$$D_{j-\frac{1}{2}} \le f'(q)\cdot\frac{3}{2} \le \max_{\dots}\left|f'(q)\right|\cdot\frac{3}{2}$$

$$0 \le 1 - \lambda D_{j-\frac{1}{2}} \ge 1 - \frac{3}{2}\lambda \max|f'| \Leftarrow \lambda \max|f'| \le \frac{2}{3}$$

What about Time Integration?

SSPRK(2,2)

$TV(u_{\ell+1})$

$$u^{(1)} = u_\ell + h_t L(u_\ell), \qquad u_{\ell+1} = \frac{u_\ell}{2} + \frac{1}{2}(u^{(1)} + h_t L(u^{(1)})).$$

Above: A version of RK2 with L the ODE RHS. Will this cause wrinkles?

$\rightarrow TV(\alpha\vec{u} + (1-\alpha)\vec{v}) \le \alpha TV(\vec{u}) + (1-\alpha)TV(\vec{v}) \qquad (0 \le \alpha \le 1)$

$$TV(u_{\ell+1}) = TV\left(\frac{u_\ell}{2} + \frac{1}{2}(u^{(1)} + h_t L(u^{(1)}))\right)$$

$$\le \frac{1}{2}TV(u_\ell) + \frac{1}{2}TV(u^{(1)} + h_t L(u^{(1)}))$$

$$\le \frac{1}{2}TV(u_\ell) + \frac{1}{2}TV(u_\ell + h_t L(u_\ell))$$

$$\le \frac{1}{2}TV(u_\ell) + \frac{1}{2}TV(u_\ell) = TV(u_\ell)$$

Total Variation is Convex

Show: TV(·) is a convex functional.

$0 \le \alpha \le 1$

$$TV(\alpha \vec{u} + (1-\alpha)\vec{v})$$

$$\le \sum_j \left| \alpha (u_j - u_{j-1}) + (1-\alpha)(v_j - v_{j-1}) \right|$$

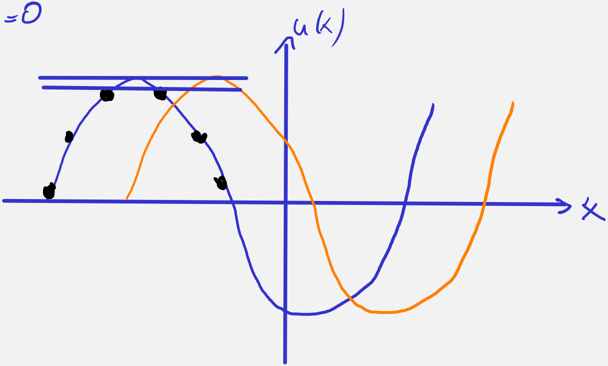
$$\le \alpha \underbrace{\sum_j |u_j - u_{j-1}|}_{TV(u)} + (1-\alpha) \underbrace{\sum_j |v_j - v_{j-1}|}_{TV(v)}$$

TVD and High Order

Can TVD schemes be high order everywhere?

$u_t + u_x = 0$

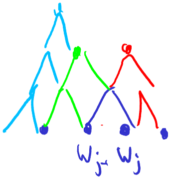
u_x^2



Near smooth extrema, TVD schemes are still only first order.

High Order at Smooth Extrema

- TVB Schemes [Shu '87]
- ENO [Harten/Engquist/Osher/Chakravarthy '87]
  - Define $W_j = w(x_{j+1/2}) = \int_{x_{1/2}}^{x_{j+1/2}} u(\xi, t) d\xi = h_x \sum_{i=1}^{j} \bar{u}_i$
    - Observe $u_{j+1/2} = w'(x_{j+1/2})$.
    - Approximate by interpolation/numerical differentiation.
  - Start with the linear function $p^{(1)}$ through W_{j-1} and W_j
  - Compute divided differences on (W_{j-2}, W_{j-1}, W_j)
  - Compute divided differences on (W_{j-1}, W_j, W_{j+1})
  - Use the one with the smaller magnitude to extend $p^{(1)}$ to quadratic
  - (and so on, adding points on the side with the lowest magnitude)

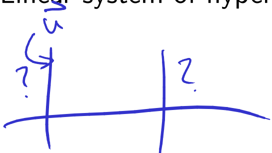


- WENO [Liu/Osher/Chan '94]

Outline

Systems of Conservation Laws

Linear system of hyperbolic conservation laws, $A \in \mathbb{R}^{m\times m}$:



$$\begin{aligned} \boldsymbol{u}_t + A\boldsymbol{u}_x &= 0, \\ \boldsymbol{u}(x,0) &= \boldsymbol{u}_0(x). \end{aligned}$$

$u_{tt} = u_{xx}$

$$\begin{aligned} u_t &= v_x \\ v_t &= u_x \end{aligned}$$

$\to u_{tt} = v_{xt} = u_{xx}$

Assumptions on A?

hyperbolic if A diagonalizable, real eigenvalues

$A\vec{r}_p = \lambda_p \vec{r}_p \quad (p = 1 \dots m)$

Called strictly hyperbolic if λ_p are all distinct.

$\vec{v} = R^{-1}\vec{u} \quad \rightsquigarrow \vec{v}_t + \Lambda \vec{v}_x = \vec{0}$

"characteristic variables"

$AR = R\Lambda$

Linear System Solution

$$\boldsymbol{v} = R^{-1}\boldsymbol{u}, \qquad \boldsymbol{v}_t + \Lambda \boldsymbol{v}_x = 0.$$

Write down the solution.

$$u(x,t) = \sum_p r_p v_p(x - \lambda_p t)$$

$$\vec{v}(x) = R^{-1}\vec{u}(x,0)$$

What is the impact on boundary conditions? E.g. $(\lambda_p) = (-c, 0, c)$ for a BC at $x = 0$ for $[0, 1]$?

v_3

Can only specify a BC on v_3.

Characteristics for Systems (1/2)

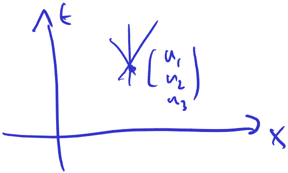
Consider system $\boldsymbol{u}_t + f(\boldsymbol{u})_x = 0$. Write in quasilinear form:

$$u_t + A(u)\, u_x = 0 \qquad A(u) = f'(u)$$

When hyperbolic?

$A(u)$ diag. w/ real eigv.
strictly if eigv. are distinct.
} only local

Characteristics for Systems (2/2)



What about characteristics/shock speeds?

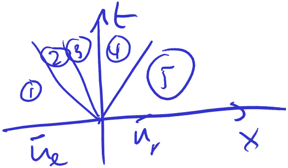
- Considering char. var.: "characteristic" still makes sense, m characteristics through (x, t)
- char speeds: not an ODE

Are values of $\boldsymbol{u}$ still constant along characteristics?

no, but CV are.

Shocks and Riemann Problems for Systems

$$
\begin{aligned}
\boldsymbol{u}_t + A\boldsymbol{u}_x &= 0, \\
\boldsymbol{u}(x,0) &= \begin{cases} \boldsymbol{u}_l & x < 0, \\ \boldsymbol{u}_r & x > 0. \end{cases}
\end{aligned}
$$



Solution? (Assume strict hyperbolicity with $\lambda_1 < \lambda_2 < \cdots < \lambda_m$.)