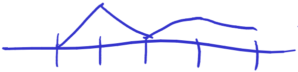
Today

- systems / 2D / 3D
- functional analysis
- elliptic problems
- FE approximation



$-u'' = f$

2

Announcements

HW4

Project

Book updates

Supplementary books

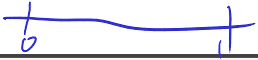
Live lecture experiment

Systems of Conservation Laws

Linear system of hyperbolic conservation laws, $A \in \mathbb{R}^{m\times m}$:

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{u}_t + A\boldsymbol{u}_x &= 0, \\ \boldsymbol{u}(x,0) &= \boldsymbol{u}_0(x). \end{aligned}$$



Assumptions on A?

A diag. / real λ_p

$$A r_p = \lambda_p r_p \quad (p = 1 \ldots m)$$

$$v = R^{-1} u \qquad AR = R\Lambda$$

$$v_t + \Lambda v_x = 0$$

Linear System Solution

$$\boldsymbol{v} = R^{-1}\boldsymbol{u}, \qquad \boldsymbol{v}_t + \Lambda \boldsymbol{v}_x = 0.$$

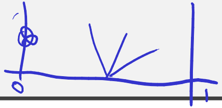
Write down the solution.

$$\vec{v}(x,0) = R^{-1}\vec{u}(x,0$$

$$\vec{u}(x,t) = \sum_p \vec{r}_p \, v_p(x - \lambda_p t, 0)$$

What is the impact on boundary conditions? E.g. $(\lambda_p) = (-c, 0, c)$ for a BC at $x = 0$ for $[0,1]$?

v_3



Characteristics for Systems (1/2)

Consider system $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0$. Write in quasilinear form:

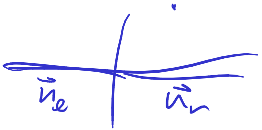
$$\vec{u}_t + A(\vec{u})\,\vec{u}_x = 0 \qquad A(u) = J_f$$

When hyperbolic?

Shocks and Riemann Problems for Systems

$$\begin{aligned} \boldsymbol{u}_t + A\boldsymbol{u}_x &= 0, \\ \boldsymbol{u}(x,0) &= \begin{cases} \boldsymbol{u}_l & x < 0, \\ \boldsymbol{u}_r & x > 0. \end{cases} \end{aligned}$$

$\vec{v}_t + \Lambda \vec{v}_x = 0$



Solution? (Assume strict hyperbolicity with $\lambda_1 < \lambda_2 < \cdots < \lambda_m$.)

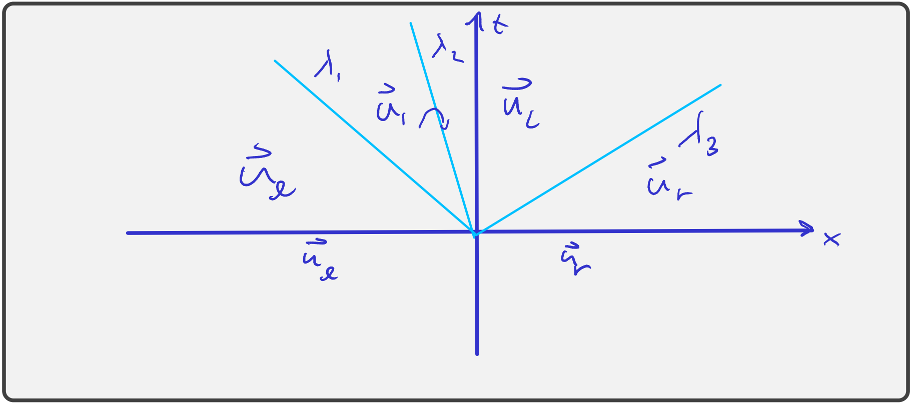
$\vec{u}_l = \sum_{p=1}^{m} \alpha_p \vec{r}_p$ $\qquad$ $\vec{u}_r = \sum_{p=1}^{m} \beta_p \vec{r}_p$ $\qquad$ $v_p(x,0) = \begin{cases} \alpha_p & x < 0, \\ \beta_p & x > 0. \end{cases}$

$P(x,t)$ is the max value of p so that $x - \lambda_p t > 0$

$$\vec{u}(x,t) = \sum_{p=1}^{P(x,t)} \beta_p \vec{r}_p + \sum_{p=P(x,t)+1}^{m} \alpha_p \vec{r}_p$$

Shock Fans (1/2)

What does the solution look like?



Jump across the characteristic associated with λ_p?

$$[\vec{u}] = (\beta_p - \alpha_p)\,\vec{r}_p$$

Shock Fans (2/2)

Do those jumps satisfy Rankine-Hugoniot?

system $\rightarrow [f] = A[u] = A(\beta_p - \alpha_p)\vec{r}_p = \lambda_p (\beta_p - \alpha_p)\vec{r}_p = \lambda_p [u]$

$\rightsquigarrow \frac{[f]}{[u]} = \lambda_p \leftarrow$ scalar only

How can we find intermediate values of $\boldsymbol{u}$?

$\vec{u}_r - \vec{u}_l = (\beta_1 - \alpha_1)\vec{r}_1 + \dots + (\beta_m - \alpha_m)\vec{r}_m$

$\oplus$ Rankine-Hugoniot

Two Dimensions

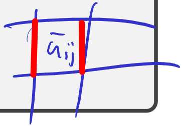
$u_t + \nabla \cdot \vec{f}(u) = 0$ $\qquad \vec{f} = \begin{pmatrix} f \\ g \end{pmatrix}$

$u_t + f(u)_x + g(u)_y = 0$. Finite volume methods generalize in principle:

$$\frac{d\bar{u}_{ij}}{dt} + \frac{1}{h^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f(u(x_{i+\frac{1}{2}}, y, t)) - f(u(x_{i-\frac{1}{2}}, y, t))\, dy$$

$$+ \frac{1}{h^2}\int$$

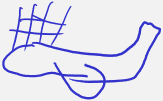
stencil not a star, but a brick



However:

- TVD in 2D ⇒ first order
- reconstruction can be expensive

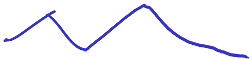
discontinuous Galerkin



Outline

Outline

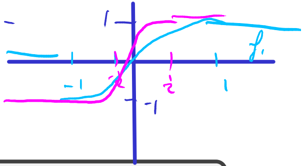
Function Spaces



Consider

$f_n \in C^1$

$$f_n(x) = \begin{cases} \cancel{0}\ -1 & x \le -\frac{1}{n}, \\ \frac{3n}{2}x - \frac{n^3}{2}x^3 & -\frac{1}{n} < x < \frac{1}{n}, \\ 1 & x \ge 1/n. \end{cases}$$



Converges to the step function. Problem?

Would like the limit to retain features (like cont.) of the sequence.

Norms

Definition (Norm)

A norm $\|\cdot\|$ maps an element of a *vector space* into $[0, \infty)$. It satisfies:

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Convergence

Definition (Convergent Sequence)

$x_n \to \underline{\underline{x}} :\Leftrightarrow \|x_n - x\| \to 0$ (convergence in norm)

Definition (Cauchy Sequence)

For all $\varepsilon > 0$, there exists an $n \geq 0$ so that $\|x_\nu - x_\mu\| < \varepsilon$

if $\mu, \nu \geq n$.

Banach Spaces

Definition (Complete/"Banach" space)

Cauchy ⇒ norm conv.

What's special about Cauchy sequences?

gives limits

Counterexamples?

$(\mathbb{Q}, |\cdot|)$

$(C^0, \|f\| := \int |f|)$

More on C^0

Let Ω $\subset \mathbb{R}^n$ be open. Is $C^0(\Omega)$ with $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$ Banach?

$\Omega = (0,1)$ $\quad f(x) = \frac{1}{x}$ $\quad f \in C^0(\Omega)$

$\|f\| = \infty$ $\qquad$ not even normed.

Is $C^0(\bar{\Omega})$ with $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$ Banach?

Assume (f_i) Cauchy. $\qquad (\mathbb{R}, |\cdot|)$

- $x \in \mathbb{R}$: $(f_i(x)) \to f$