Today

- Elliptic PDEs via Sobolev
  - "Yukawa" $-\Delta u + h = f$
  - Poisson $-\Delta u = f$ — Riesz
    - Lax-Milgram
    - Poincaré ineq.
- Finite dim approx

Announcements

- Feedback
- Jitsi improved
- HW4 due Friday

An Elliptic Model Problem

Let $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H^1(\Omega)$.

$$\begin{aligned} -\nabla \cdot \nabla u + u &= f(x) \quad (x \in \Omega), \\ u(x) &= 0 \quad (x \in \partial\Omega). \end{aligned}$$

Let $V := H_0^1(\Omega)$. Integration by parts? (Gauss's theorem applied to $a\boldsymbol{b}$):

Weak form?

Let $v \in H_0^1(\Omega)$

$$\int -\nabla \cdot \nabla u \cdot v + \int uv = \int fv$$

$$= \int_\Omega \nabla u \cdot \nabla v - \cancel{\int_{\partial\Omega} v\, \partial_n u} + \int_\Omega uv + \int fv$$

Motivation: Bilinear Forms and Functionals

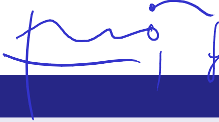
$$\overbrace{\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv}^{a(u,v)} = \overbrace{\int fv}^{g(v)}.$$

This is the weak form of the strong-form problem. The task is to find a $u \in V$ that satisfies this for all test functions $v \in V$.

Recast this in terms of bilinear forms and functionals:

$$a(u,v) = g(v)$$

Dual Spaces and Functionals



$f(x+y) = f(x) + f(y)$

Bounded Linear Functional

Let $(V, \|\cdot\|)$ be a Banach space. A linear functional is a linear function $g : V \to \mathbb{R}$. It is bounded ($\Leftrightarrow$ continuous) if there exists a constant C so that $|g(v)| \leq C \|v\|$ for all $v \in V$.

$\alpha g(v) + \beta h(v) = k(v)$

Dual Space

Let $(V, \|\cdot\|)$ be a Banach space. Then the dual space V' is the space of bounded linear functionals on V.

Dual Space is Banach (cf. e.g. Trèves 1967)

V' is a Banach space with the dual norm

$$\|g\|_{V'} = \sup_{v \in V \setminus \{0\}} \frac{|g(v)|}{\|v\|_V}$$

Functionals in the Model Problem

$a(u,v) = \dots$
$g(v) = \int fv = (f,v)_{L^2}$ $\Big\}\; v \in H_0^1$

Is g from the model problem a bounded functional? (In what space?)

$$\underbrace{\|g\|_{V'} = \sup_{v \in H_0^1 \setminus \{0\}} \frac{|(f,v)_{L^2}|}{\|v\|_{H^1}}} \le \frac{\|f\|_{L^2}\|v\|_{L^2}}{\|v\|_{L^2} + \underbrace{\|D_x v\|_{L^2}}} \le \frac{\|f\|_{L^2}\cancel{\|v\|_{L^2}}}{\cancel{\|v\|_{L^2}}}$$

That bound felt loose and wasteful. Can we do better?

$$\|f\|_{H^{-1}} = \sup_{v \in H^1(\Omega)\setminus\{0\}} \frac{|\langle f,v\rangle|}{\|v\|_{H^1}}$$

$\|f\|_{H^{-1}} < \infty \;\Leftrightarrow:\; f \in H^{-1}$

Riesz Representation Theorem (1/3)

$a(u,v) = (u,v)_{H^1}$

Let V be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Theorem (Riesz)

Let g be a bounded linear functional on V, i.e. $g \in V'$. Then there exists a unique $u \in V$ so that $g(v) = \langle u, v \rangle$ for all $v \in V$.

$\hookrightarrow a(u,v)$

$g : \mathbb{R}^n \to \mathbb{R}$

Let $g \in V'$. $N(\cdot)$ nullspace.

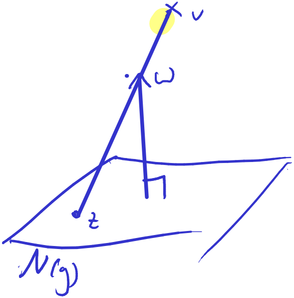
- $N(g) = V$. $u = 0$ does the job

- $N(g) \neq V$. Let $w \in N(g)^\perp$, $\alpha = g(w) \neq 0$.

$$g\left(\frac{g(v)}{\alpha} w\right) = \frac{g(v)}{g(w)} \cdot g(w) = g(v)$$

Let $v \in V$ arbitrary. $z := v - (g(v)/\alpha)\, w$.

$g(z) = 0 \Rightarrow z \in N(g)$



$z \perp w \iff (z, w) = 0$

$\left(\text{because } \begin{array}{l} z \in N(g) \\ w \in N(g)^{\perp} \end{array} \right)$

Riesz Representation Theorem: Proof (2/3)

Have $w \in N(g)^{\perp} \setminus \{0\}$, $\alpha = g(w) \neq 0$, and $z := v - (g(v)/\alpha)w \perp w$.

$$0 = \left\langle v - \frac{g(v)}{\alpha} w, w \right\rangle \Leftrightarrow \left\langle \frac{g(v)}{\alpha} w, w \right\rangle = \langle v, w \rangle \quad \Big| \cdot \frac{\alpha}{\langle w, w \rangle}$$

$$\frac{g(v)}{\cancel{\langle w, w \rangle}} \cancel{\langle w, w \rangle} = \left\langle v, \underbrace{\frac{w \cdot \alpha}{\langle w, w \rangle}}_{u :=} \right\rangle$$

$$g(v) = \langle v, u \rangle$$

Riesz Representation Theorem: Proof (3/3)

Uniqueness of u?

Suppose u and $\hat{u}$ are Riesz representers.

$$g(v) = \langle u, v \rangle = \langle \hat{u}, v \rangle \Rightarrow 0 = \langle \underbrace{u - \hat{u}}_{=0}, v \rangle$$

for all $v \in V$.

Back to the Model Problem

$$\begin{aligned} a(u,v) &= \langle \nabla u, \nabla v\rangle_{L^2} + \langle u, v\rangle_{L^2} = \langle u, v\rangle_{H^1} \\ g(v) &= \langle f, v\rangle_{L^2} \\ \langle u, v\rangle_{H^1} = a(u,v) &= g(v) \end{aligned}$$

Have we learned anything about the solvability of this problem?

Riesz rep. of g wrt H^1 inner prod gives u.

Poisson

Let $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H^{-1}(\Omega)$.

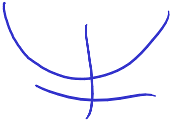
$$-\nabla \cdot \nabla u = f \quad (x \in \Omega)$$
$$u(x) = 0 \quad (x \in \partial\Omega)$$

This is called the Poisson problem (with Dirichlet BCs).

Weak form?

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx}_{a(u,v)} = \underbrace{\int f v}_{g(v)} \qquad u, v \in H^1_0(\Omega)$$

Ellipticity



$a(u,v) = g(v)$

Let V be Hilbert space.

V-Ellipticity

A bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called coercive if there exists a constant $c_0 > 0$ so that

$$c_0 \|u\|_V^2 \le a(u,u) \quad \text{for all } u \in V$$

and a is called continuous if there exists a constant $c_1 > 0$ so that

$$|a(u,v)| \le c_1 \|u\|_V \|v\|_V .$$

If a is both coercive and continuous on V, then a is said to be V-elliptic.

Lax-Milgram Theorem

Let V be Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

avoidable

Lax-Milgram, Symmetric Case

Let a be a V-elliptic bilinear form that is also symmetric, and let g be a bounded linear functional on V.
Then there exists a unique $u \in V$ so that $a(u, v) = g(v)$ for all $v \in V$.

a still defines an inner product.

- $\langle u,u\rangle_a = a(u,u) \geq c\|u\|_V^2 > 0$ for all u

coercivity.

- $0 = \langle u,u\rangle_a = a(u,u) \geq c_0 \|u\|_V^2 \geq 0$

$\Rightarrow \|u\|^2 = 0 \Rightarrow u = 0$

$\Rightarrow \langle \cdot, \cdot \rangle_a$ is an IP. $\Rightarrow$ Riesz. rep. wrt. gives $\exists$ and uniqueness!

Back to Poisson

$$\|\nabla u\|_{L^2} \leq \|u\|_{L^2} + \|\nabla u\|_{L^2} = \|u\|_{H^1}$$

Can we declare victory for Poisson?

$$|a(u,v)| = \left|\int \nabla u \cdot \nabla v\right| = |(\nabla u, \nabla v)_{L^2}| \leq \|\nabla u\|_{L^2} \cdot \|\nabla v\|_{L^2} \leq \|u\|_{H^1} \cdot \|v\|_{H^1}$$

$$a(u,u) = \int \nabla u \cdot \nabla u \overset{?}{\geq} c_0 \left(\int \nabla u \cdot \nabla u + \int u^2 \right) = \|u\|_{H^1}$$

Can this inequality hold in general, without further assumptions?

Can't work for constants