Today

- $-\Delta u = f$
  - Poincaré's ineq.
- $a(u,v) = \ell(v) \quad \forall v \in V$
  - $\hookrightarrow a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h$

    $\|u - u_h\|_{H^1} \leftarrow$ Céa's Lemma

  $\|u - u_h\|_{L^2} \leftarrow$ Aubin-Nitsche Lemma

Announcements

- HW deadline $\rightarrow$ Apr. 15 (next Wed)
- proj. deadline $\rightarrow$ April 22
- assignments for rest of semester
- Firedrake install

Poisson

Let $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H^{-1}(\Omega)$.

$$-\nabla \cdot \nabla u = f$$
$$u(x) = 0 \quad \text{on } \partial\Omega$$

This is called the Poisson problem (with Dirichlet BCs).

Weak form?

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx}_{a(u,v)} = \underbrace{\int f(x)\, v(x)\, dx}_{g(v)} \quad \forall v \in H^1_0$$

$$u \in H^1_0$$

Ellipticity

Let V be Hilbert space.

V-Ellipticity

A bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called coercive if there exists a constant $c_0 > 0$ so that

$$c_0 \|u\|_V^2 \le a(u,u)$$

and a is called continuous if there exists a constant $c_1 > 0$ so that

$$|a(u,v)| \le c_1 \|u\|_V \|v\|_V$$

If a is both coercive and continuous on V, then a is said to be V-elliptic.

Lax-Milgram Theorem

Let V be Hilbert space with inner product $\langle \cdot, \cdot \rangle$. $a(u,v) = a(v,u)$

Lax-Milgram, Symmetric Case

Let a be a V-elliptic bilinear form that is also symmetric, and let g be a bounded linear functional on V.
Then there exists a unique $u \in V$ so that $a(u, v) = g(v)$ for all $v \in V$.

Back to Poisson

Can we declare victory for Poisson?

$$\left|\int_\Omega \nabla u \cdot \nabla v \, dx\right| = |\langle \nabla u, \nabla v \rangle_{L^2}| \leq \|\nabla u\|_{L^2} \|\nabla v\|_2 \leq \|u\|_{H^1} \|v\|_{H^1}$$

$$\int \nabla u \cdot \nabla u \, dx \geq c_0 \left(\int_\Omega \nabla u \cdot \nabla u + \int u^2 \right)$$

$$\|u\|_{L^2} \leq C \cdot \|\nabla u\|_{L^2}$$

Can this inequality hold in general, without further assumptions?

constant violates that

$u \in H^1_0$

Poincaré-Friedrichs Inequality (1/3)

Theorem (Poincaré-Friedrichs Inequality)

Suppose $\Omega \subset \mathbb{R}^n$ is bounded and $u \in H_0^1(\Omega)$. Then there exists a constant $C > 0$ such that

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}.$$

$u^2(\vec{x}) \cdot \vec{x}$

$$\nabla \cdot (u^2 \vec{x}) = \partial_{x_1}(u^2 x_1) + \cdots \partial_{x_n}(u^2 x_n)$$

$$= u^2 + 2(u\, \partial_{x_1} u) + \cdots + u^2 + 2(u\, \partial_{x_n} u)$$

$$= n u^2 + 2u(\nabla u \cdot x)$$

$$u^2 = \frac{1}{n} \cdot \nabla \cdot (u^2 x) - \frac{2}{n} u(\nabla u \cdot x)$$

Poincaré-Friedrichs Inequality (2/3)

Prove the result in $C_0^\infty(\Omega)$.

$$\|u\|_{L^2} \le \overset{C}{} \|\nabla u\|_{L^2}$$

$$\|u\|_{L^2}^2 = \int_\Omega u^2 = \int_\Omega \frac{1}{n} \nabla \cdot (u^2 x) - \frac{2}{n} u(\nabla u \cdot x)\,dx$$

$$= \cancel{\frac{1}{n} \int_{\partial\Omega} \hat{n} \cdot (u^2 x)} \, dS_x - \frac{2}{n} \int u (\nabla u \cdot x)\,dx$$

$u \in H_0^1$

$$\le \frac{2}{n} \max_{x \in \Omega} |x|_2 \int_\Omega |u \nabla u|\,dx \le \underbrace{\frac{2}{n} \max_{x \in \Omega} |x|}_{C} \|u\|_{L^2} \cdot \|\nabla u\|_{L^2}$$

$$\Rightarrow \|u\|_2 \le C \cdot \|\nabla u\|_{L^2}$$

Poincaré-Friedrichs Inequality (3/3)

Prove the result in $H_0^1(\Omega)$.

Let $u \in H_0^1(\Omega)$. $(u_k) \subset C_0^\infty(\Omega)$ so that $\|u_k - u\|_{H^1} \to 0$.
Then the inequality holds for each u_k.

By continuity, the inequality also holds for u.

Back to Poisson, Again

Show that the Poisson bilinear form is coercive. $\le a(u,u)$

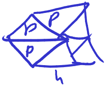
$$\frac{1}{C^2+1}\|u\|_{H^1}^2 = \frac{1}{C^2+1}\Big(\underbrace{\|u\|_{L^2}^2}_{\le C^2\|\nabla^w u\|_{L^2}^2} + \|\nabla^w u\|_{L^2}^2\Big) \le \|\nabla^w u\|_{L^2}^2 = a(u,u).$$

Draw a conclusion on Poisson:

Because of coercivity and continuity, Poisson has a unique solution in $H_0^1(\Omega)$!

Outline

Ritz-Galerkin



$h \to 0$

$\|u - u_h\| \leq h^{?} \, \|\cdots\|$

Some key goals for this section:

- How do we use the weak form to compute an approximate solution?
- What can we know about the accuracy of the approximate solution?

Can we pick one underlying principle for the construction of the approximation?

Hilbert

$a(u,v) = g(v) \quad v \in V \qquad V \subseteq H$

Choose a finite-dim subspace $V_h \subset V$. Find a solution $u_h \in V_h$

$$a(u_h, v_h) = g(v_h) \quad v_h \in V_h$$

u_h is called the Ritz-Galerkin approximation.

Galerkin Orthogonality

$$a(u, v) = g(v) \qquad \text{for all } v \in V, a(u_h, v_h) \quad = g(v_h) \qquad \text{for all } v_h \in V_h.$$

Observations?

$\ominus$

$$a(u, v_h) = g(v_h) \quad v_h \in V_h$$

$$\Rightarrow a(\underbrace{u - u_h}_{\text{approx error}}, v_h) = 0$$

"Galerkin orthogonality"

Céa's Lemma

Let $V \subset H$ be a closed subspace of a Hilbert space H.

Céa's Lemma

Let $a(\cdot,\cdot)$ be a coercive and continuous bilinear form on V. In addition, for a bounded linear functional g on V, let $u \in V$ satisfy

$$a(u, v) = g(v) \qquad \text{for all } v \in V.$$

Consider the finite-dimensional subspace $V_h \subset V$ and $u_h \in V_h$ that satisfies

$$a(u_h, v_h) = g(v_h) \qquad \text{for all } v_h \in V_h.$$

Then

$$\boxed{\|u - u_h\|_V \leq \frac{c_1}{c_0} \inf_{v_h \in V_h} \|u - v_h\|}$$

Céa's Lemma: Proof

Recall Galerkin orthgonality: $a(u_h - u, v_h) = 0$ for all $v_h \in V_h$. Show the result.

Let $v_h \in V_h$

$$c_0 \|u - u_h\|_V^2 \le a(u-u_h, u-u_h) \quad \text{(coerciv.)}$$

$$= a(u-u_h, u-v_h) + \cancel{a(\underbrace{u-u_h}_{\text{error}}, v_h - u_h)} \quad \perp_a V_h$$

$$= a(u-u_h, u-v_h) \le c_1 \|u-u_h\|_V \|u-v_h\|_V .$$

Elliptic Regularity

→ $m=1$ for now → $s=2$

Definition (H^s Regularity)

Let $m \geq 1$, $H_0^m(\Omega) \subseteq V \subseteq H^m(\Omega)$ and $a(\cdot,\cdot)$ a V-elliptic bilinear form. The bilinear form $a(u,v) = \langle f, v \rangle$ for all $v \in V$ is called H^s regular, if for every $f \in H^{s-2m}$ there exists a solution $u \in H^s(\Omega)$ and we have with a constant $C(\Omega, a, s)$,

$-u'' = f$

$$\rightarrow \underbrace{\|u\|_{H^s}}_{H^2} \leq C(\Omega, s, a) \underbrace{\|f\|_{H^{s-2m}}}_{L^2}$$

→ Gilbarg Trudinger.

Theorem (Elliptic Regularity (cf. Braess Thm. 7.2))

Let a be a H_0^1-elliptic bilinear form with sufficiently smooth coefficient functions.

For Poisson

- If Ω convex, then the Dirichlet problem is H^2 regular
- Let $s \geq 2$. If $\partial\Omega$ is C^s, then the P. Dirichlet prob. is H^s reg.

Elliptic Regularity: Counterexamples

Are the conditions on the boundary essential for elliptic regularity?

Are there any particular concerns for mixed boundary conditions?

Estimating the Error in the Energy Norm

Come up with an idea of a bound on $\|u - u_h\|_{H^1}$.

$$\|u-u_h\|_{H^1} \underset{\text{Céa}}{\leq} C \inf_{v \in V_h} \|u-v_h\|_{H^1} \leq C \|u - I_h u\|_{H^1}$$

$$\underset{?}{\leq} C' h \|u\|_{H^2} \leq C' h \cdot C(\Omega, a, f) \|f\|_{L^2}$$

What's still to do?