Today

- L^2 estimate
  - ↳ Aubin - Nitsche
  - ↳ elliptic regularity
- FEM assembly 1D
- FEM 2D
- FEM approximation
- Mixed FEM

Announcements

- HW4 due
- HW5 out soon
- Project due in a week

$$\|u-u_h\|_{H^1} \leq \inf_{v \in V_h} \|u-v_h\|_{H_1} \leq \|u - I_h u\|_{H_1} \leq ?!$$

Céa's Lemma

Let $V \subset H$ be a closed subspace of a Hilbert space H.

Céa's Lemma

Let $a(\cdot,\cdot)$ be a coercive and continuous bilinear form on V. In addition, for a bounded linear functional g on V, let $u \in V$ satisfy

$$a(u, v) = g(v) \qquad \text{for all } v \in V.$$

Consider the finite-dimensional subspace $V_h \subset V$ and $u_h \in V_h$ that satisfies

$$a(u_h, v_h) = g(v_h) \qquad \text{for all } v_h \in V_h.$$

Then

$$\boxed{\|u - u_h\| \le \frac{c_1}{c_0} \inf_{v_h \in V_h} \|u - v_h\|}$$

Elliptic Regularity

Definition (H^s Regularity)

Let $m \geq 1$, $H_0^m(\Omega) \subseteq V \subseteq H^m(\Omega)$ and $a(\cdot, \cdot)$ a V-elliptic bilinear form. The bilinear form $a(u, v) = \langle f, v \rangle$ for all $v \in V$ is called H^s regular, if for every $f \in H^{s-2m}$ there exists a solution $u \in H^s(\Omega)$ and we have with a constant $C(\Omega, a, s)$,

$$\|u\|_{H^s} \leq C(\Omega, a, s) \|f\|_{H^{s-2m}}$$

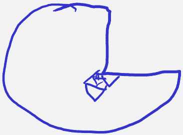
Theorem (Elliptic Regularity (cf. Braess Thm. 7.2))

Let a be a H_0^1-elliptic bilinear form with sufficiently smooth coefficient functions.

- Ω convex $\Rightarrow$ H^2-regular (Dirichlet)
- $s \geq 2$ $\partial\Omega \in C^s$ $\Rightarrow$ Dirichlet prob is H^s-regular

Elliptic Regularity: Counterexamples

Are the conditions on the boundary essential for elliptic regularity?



$\Delta u = 0$, $u(e^{i\varphi}) = \sin\left(\frac{2}{3}\varphi\right)$ on circ

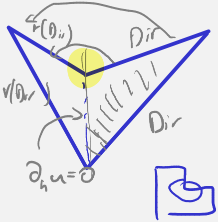
$u = 0$ otherw.

$z = x + iy$

$u(z) = \mathrm{Im}(z^{2/3})$

$u'(z) = \mathrm{Im}\left(\frac{2}{3} z^{-1/3}\right)$ not H^2 reg!

Are there any particular concerns for mixed boundary conditions?



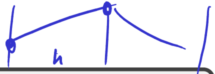
mixed BC problem

"equiv." to pure Dirichlet on bigger domain w/ reentrant cone

↪ Gilbarg / Trudinger

Estimating the Error in the Energy Norm

$$\max_{x\in[\alpha,\beta]} |(u-I_h u)(x)| \le C\cdot h^2 \cdot \|u''\|_{\infty}$$



Come up with an idea of a bound on $\|u-u_h\|_{H^1}$.

$$\|u-u_h\|_{H^1} \le \frac{c_1}{c_0} \inf_{v_h\in V_h} \|v_h-u\| \le C\|u-I_h u\|_{H^1} \overset{?}{\le} C' h\, \|u\|_{H^2}$$

$$\le C_2 h\, \|f\|_{L^2}$$

What's still to do?

- V_h ?
- I_h ?
- intp. error bound
- L^2 error

L^2 Estimates

Let H be a Hilbert space with the norm $\|\cdot\|_H$ and the inner product $\langle \cdot, \cdot \rangle$.
(Think: $H = L^2$, $V = H^1$.) $\|u\|_{L^2} \leq \|u\|_{H^1}$ ✓

Theorem (Aubin-Nitsche)

Let $V \subseteq H$ be a subspace that becomes a Hilbert space under the norm $\|\cdot\|_V$. Let the embedding $V \to H$ be continuous. Then we have for the finite element solution $u \in V_h \subset V$:

$$\|u - u_h\|_H \leq c_1 \|u - u_h\|_V \cdot \underbrace{\sup_{g \in H} \left[\frac{1}{\|g\|_H} \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V \right]}_{\text{"a}_h\text{"}}$$

if with every $g \in H$ we associate the unique (weak) solution ϕ_g of the equation (also called the dual problem)   primal $a(u, v) = \langle g, v \rangle$

$$a(w, \varphi_g) = \langle g, w \rangle \quad \text{for all } w \in V$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v = \int f v$$

$$a(w,\varphi_g) = \int_\Omega \nabla w \cdot \nabla \varphi_g = \int g w$$

$$\int u v'$$

Aubin-Nitsche: Proof

Galerkin orth: $a(u-u_h, V_h) = 0$

$$\|w\|_H = \sup_{g \in H \setminus \{0\}} \frac{\langle g, w\rangle}{\|g\|_H}$$

$$\langle g, u-u_h\rangle \underset{\text{dualp.}}{=} a(u-u_h, \varphi_g) \underset{\text{GO}}{=} a(u-u_h, \varphi_g - v_h)$$

$\hookrightarrow$ Let $v_h \in V_h$

$$\underset{\text{cont.}}{\leq} c_1 \|u-u_h\|_V \|\varphi_g - v_h\|_V$$

$$\langle g, u-u_h\rangle \leq c_1 \|u-u_h\|_V \inf_{v_h \in V_h} \|\varphi_g - v_h\|$$

$$\|u-u_h\|_H = \sup_g \frac{\langle g, u-u_h\rangle}{\|g\|}$$

✓

L^2 Estimates using Aubin-Nitsche

$$\|u - u_h\|_H \leq c_1 \|u - u_h\|_V \sup_{g\in H} \left[\frac{1}{\cancel{\|g\|_H}} \overbrace{\inf_{v_h\in V_h} \|\varphi_g - v_h\|_V}^{\leq C\cdot h \|\cancel{g}\|_H} \right],$$

If $u \in H_0^1(\Omega)$, what do we get from Aubin-Nitsche?

$$\|u-u_h\|_{L^2} \leq C\cdot h \cdot \|u-u_h\|_{H^1}$$

So does Aubin-Nitsche give us an L^2 estimate?

$$\|u-u_h\|_{L^2} \leq C' \cdot h \cdot h \cdot \|f\|_{L^2}$$

Outline

Finite Elements in 1D: Discrete Form

$-u'' = f$

$\Omega := [\alpha, \beta]$. Look for $u \in H^1_0(\Omega)$, so that $a(u, \varphi) = \langle f, \varphi \rangle$ for all $\varphi \in H^1_0(\Omega)$. Choose $V_h = \operatorname{span}\{\psi_1, \ldots, \psi_n\}$ and expand $u_h = \sum_{i=1}^{n} u_h^i \psi_i \in V_h$. Find the discrete system.

$n = n(h)$

$$a\Big(\sum_{i=1}^{n} u_h^i \psi_i, \varphi\Big) = (f, \varphi) \qquad \varphi \in \underline{\underline{V_h}}$$

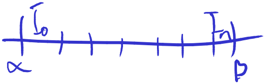
$$\Leftrightarrow \quad a\Big(\sum_{i=1}^{n} u_h^i \psi_i, \psi_j\Big) = (f, \psi_j) \qquad j = 1..n$$

$$= \sum_{i=1}^{n} u_h^i \, a(\psi_i, \psi_j) = (f, \psi_j) \qquad j = 1..n$$

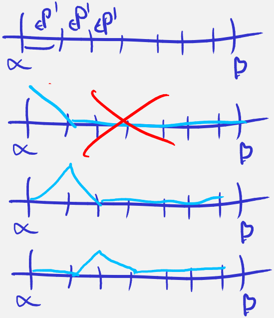
Grids and Hats

Let $I_i := [\alpha_i, \beta_i]$, so that $\bar{\Omega} = \bigcup_{i=0}^{N} I_i$ and $I_i^\circ \cap I_j = \emptyset$ for $i \neq j$. Consider a grid

$$\alpha = x_0 < \cdots < x_N < x_{N+1} = \beta,$$



i.e. $\alpha_i = x_i$, $\beta_i = x_{i+1}$ for $i \in \{0, \ldots, N\}$. The $\{x_i\}$ are called nodes of the grid. $h_i := x_{i+1} - x_i$ for $i \in \{0, \ldots, N\}$ and $h := \max_i h_i$. V_h? Basis?



$$\varphi_i(x) = \begin{cases} \frac{1}{h_{i-1}}(x - x_{i-1}) & x \in I_{i-1} \\ \frac{1}{h_i}(x_{i+1} - x) & x \in I_i \\ 0 & \text{otherwise} \end{cases}$$