Today

- 1D FE < code
- 2D FE < code

Announcements

- HW5 out this weekend
- Proj. 1 due Wed

Finite Elements in 1D: Discrete Form

$\Omega := [\alpha, \beta]$. Look for $u \in H_0^1(\Omega)$, so that $a(u, \varphi) = \langle f, \varphi \rangle$ for all $\varphi \in H_0^1(\Omega)$. Choose $V_h = \text{span}\{\varphi_1, \ldots, \varphi_n\}$ and expand $u_h = \sum_{i=1}^n u_h^i \varphi_i \in V_h$. Find the discrete system.

$$-u'' = f$$

$$a(u, v) = \langle f, v \rangle$$

$$\sum_j u_h^j \, a(\varphi_j, \varphi_i) = \langle f, \varphi_i \rangle \qquad i = 1 \ldots N$$

test fun

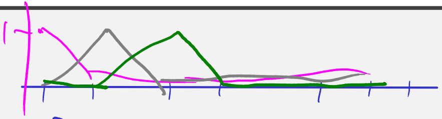
$\rightarrow$ Lows

Grids and Hats

Let $I_i := [\alpha_i, \beta_i]$, so that $\bar{\Omega} = \bigcup_{i=0}^{N} I_i$ and $I_i^\circ \cap I_j = \emptyset$ for $i \neq j$. Consider a grid

$$\alpha = x_0 < \cdots < x_N < x_{N+1} = \beta,$$

i.e. $\alpha_i = x_i$, $\beta_i = x_{i+1}$ for $i \in \{0, \ldots, N\}$. The $\{x_i\}$ are called nodes of the grid. $h_i := x_{i+1} - x_i$ for $i \in \{0, \ldots, N\}$ and $h := \max_i h_i$. V_h? Basis?



I_0

φ_0 φ_1 φ_2

$\mathbb{P}^1_h = \{ v_h \in C^0(I) : v_h|_{I_i} \in \mathbb{P}^1 \}$

Degrees of Freedom and Matrices

Define something more general than basis coefficients to solve for.

$\gamma_i : V_h \to \mathbb{R}$ ← degrees of freedom

Requirement: $(\gamma_i(v_h) = v_i)_{i=1}^{N}$ must uniquely determine $v_h \in V_h$

$\gamma_0(v_h) = f(x_0)$ $\gamma_1(v_h) = f'(x_0)$ $\gamma(v_h) = f(x)$

Now express the solve, recalling $u_h = \sum_{i=1}^{N} u_h^i \varphi_i$.

↪ in the setting of the hat functions, we have $\gamma_i(v_h) = v_h(x_i)$

"unisolvency"

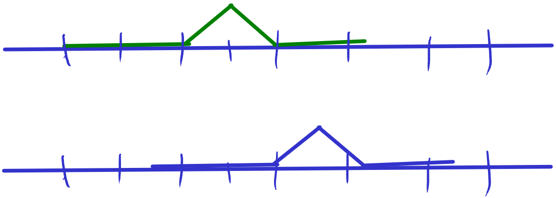
$$\sum_{j=1}^{N} \underbrace{\gamma_j(\hat{\varphi}_j)}_{u_j} \underbrace{a(\hat{\varphi}_j, \hat{\varphi}_i)}_{A_{i,j}} = \underbrace{\langle f, \hat{\varphi}_i \rangle}_{f_i}$$

Define a new basis $\hat{\varphi}_i$ so that $\gamma_i(\hat{\varphi}_j) = \delta_{i,j}$

Anything special about the matrix?

↪ in the hat functions $A_{i,i-1}$ $A_{i,i}$ $A_{i,i+1} \neq 0$ only

$a(u,v) = \int u'v'$



$\hat{\varphi}_i$

$\hat{\varphi}_j$

$\rightarrow$ sparse!

Error Estimation

According to Céa, what's our main missing piece in error estimation now?

$$I_h^1 : C^0(\bar{\Omega}) \to P_h^1$$

$$v \longmapsto \sum_{i=1}^{N} \gamma_i(v)\, \hat{\varphi}_i \in P_h^1$$

Interpolation Error (1D only)

For $v \in H^2(\Omega)$,

$$\| v - I_h v \|_{L^2} \leq C \underbrace{h^2} \| D_w^2 v \|_{L^2}$$

$$\| D_w (v - I_h v) \|_{L^2} \leq C \underbrace{h} \| D_w v \|_{L^2}$$

If $v \in H^1(\Omega) \setminus H^2(\Omega)$,

$$\| v - I_h v \|_{L^2} \leq C h \| D_w^2 v \|_{L^2}$$

$$\lim_{h \to 0} \| D_w (v - I_h v) \|_{L^2} \to 0$$

Is I_h^1 defined for $v \in H^2$? for $v \in H^1 \setminus H^2$?

$H^2 \to C^0$ is continuous (dep. on domain, dim...)

Sobolev embedding thm

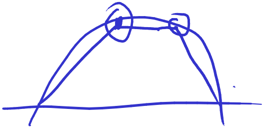
Interpolation Error: Towards an Estimate

Provide an a-priori estimate.

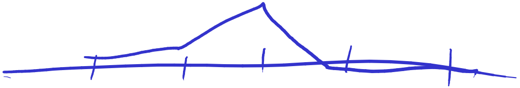
$$\|u-u_h\|_{H^1} \le \frac{c_1}{c_0} \inf_{v_h \in P^1_h} \|u-v_h\|_{H^1} \le \frac{c_1}{c_0} \|u - I^1_h u\|_{H^1}$$
$$\le C h \|D_x^2 u\|_{L^2}.$$

What's the relationship between $I^1_h u$ and u_h?

None.



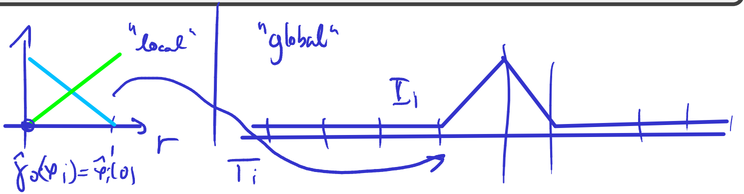
Local-to-Global



Is there a simple way of constructing the polynomial basis?

L2G map: $T_i : \hat{E} \to I_i$

$\hat{E} = [0,1]$



Local-to-Global: Math

Construct a polynomial basis using this approach.

$$\hat{\varphi}_0(r) = 1 - r$$

$$\hat{\varphi}_1(r) = r$$

$$\varphi_i(x) = \begin{cases} (\hat{\varphi}_1 \circ T_{i-1}^{-1})(x) & x \in I_{i-1} \\ (\hat{\varphi}_0 \circ T_i)(x) & x \in I_i \end{cases}$$

Demo

Demo: Developing FEM in 1D

Going Higher Order

Possible extension:

$$P_h^k = \{ v_h \in C^0(\bar{\Omega}) \; : \; v_h|_{I_i} \in P^k \}$$

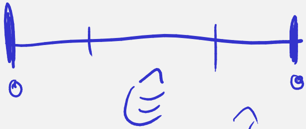
Higher Order Approximation

Let $0 \leq \ell \leq k$. Then for $v \in H^{\ell+1}(\Omega)$,

$$\| v - I_h^k v \|_{L^2} + \| D_w (v - I_h^k v) \|_{L^2} \leq C \cdot h^{\ell+1} \| D_w^{\ell+1} v \|_{L^2}.$$

High-Order: Degrees of Freedom

Define some degrees of freedom (or DoFs) for high-order 1D FEM.



Lagrange basis on
each element