CS555

$$-\varepsilon \nabla \cdot \nabla u + b \cdot \nabla u = f$$

$$u = 0 \text{ on } \Gamma$$

Galerkin form:

$$-\varepsilon(\nabla u, \nabla v) + (b \cdot \nabla u, v) = (f, v)$$

1. non-symmetric
2. coercive?

<u>Options</u>

1. Pick a better F.E. space
2. "stabilize" add a term to Galerkin (SUPG)
3. least-squares

Objectives

1. Outline the basic "mechanics" of a least-squares method
2. Identify drawbacks
3. Construct a L.S. method and build a connection the norm (for Lax-Milgram)
4. Introduce an "error estimate"

Recall

Take $Lu = f$ on Ω

some operator

Galerkin: Find $u \in V$ s.t.

$$\underbrace{(Lu, v)}_{\text{let } a(u,v) \text{ or IBP}} = (f, v) \quad \forall v \in V$$

Show $$\underbrace{c_0 \|u\|_V^2 \leq a(u,u) \leq c_1 \|u\|_V^2}_{a(\cdot,\cdot) \text{ is } V\text{-norm equivalent.}}$$

Let's assume L is non-singular

example

$$L = -u'' \rightarrow \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ & & & -1 & 1 \end{bmatrix}$$

$$\rightarrow L\,\underline{1} = 0$$

$L = -u''$ with $u(0)=0$
$$\rightarrow \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

$$\text{Let } G(u;f) = \| Lu - f \|_0^2$$

$= $ "the least squares L^2 functional"

$$= \langle Lu - f, \; Lu - f \rangle_0$$

$G(u; f) = \|Lu - f\|^2$

Q: what minimize $G(u)$?

The Gâteaux Derivative:

let $f: U \to V$

The G-derivative of f at u in the direction of v is

$$d_v G(u) = \lim_{\varepsilon \to 0} \frac{G(u + \varepsilon v) - G(u)}{\varepsilon}$$

Set $d_v G(u) = 0 \ \forall v$, find u.

$$G(u) = \|Lu - f\|^2$$

$$G(u+\varepsilon v) = \langle Lu + \varepsilon Lv - f,\ Lu + \varepsilon Lv - f \rangle$$

$$= \langle Lu - f,\ Lu - f \rangle + 2\varepsilon \langle Lu - f, Lv \rangle + \varepsilon \langle Lv, Lv \rangle$$

$$\rightarrow \frac{G(u+\varepsilon v) - G(u)}{\varepsilon} = \frac{\cancel{\langle Lu-f, Lu-f\rangle} + 2\cancel{\varepsilon}\langle Lu-f, Lv\rangle + \varepsilon^{\cancel{2}}\langle Lv, Lv\rangle - \cancel{\langle Lu-f, Lu-f\rangle}}{\varepsilon}$$

$$= 2\langle Lu - f, Lv\rangle + \varepsilon\langle Lv, Lv\rangle$$

$$\xrightarrow[\varepsilon \to 0]{} = 2\langle Lu - f, Lv \rangle$$

$\rightarrow$ set $\equiv 0 \ \forall v:$

$$2\langle Lu - f, Lv\rangle = 0 \quad \forall v \in V$$

$$\rightarrow \boxed{\langle Lu, Lv\rangle = \langle f, Lv\rangle}$$

Similar in $\mathbb{R}^n$

$\underline{\underline{A}}$, $\underline{b}$

$$\min \|Ax - b\|$$

x satisfies

$$A^T A x = A^T b$$

x satisfies

$$\langle A^T A x, v\rangle = \langle A^T b, v\rangle \quad \forall v \in \mathbb{R}^n$$

$$\rightarrow \langle Ax, Av\rangle = \langle b, Av\rangle$$

Let $V^h \subset V$. Then this holds:

$$\min_{u^h \in V^h} \| L u^h - f \|_0 \longrightarrow \langle L u^h, L v \rangle = \langle f, L v \rangle$$

$$\forall\, v \in V^h \subset V.$$

If ϕ_i is a basis for V^h.
(a "hat" function)

Then let $u^h = \sum \alpha_i \phi_i$

$$A_{ij} := \langle L\phi_j, L\phi_i \rangle$$

$$b_i = \langle f, L\phi_i \rangle$$

A few observations

① $A_{ij} = \langle L\phi_j, L\phi_i \rangle$

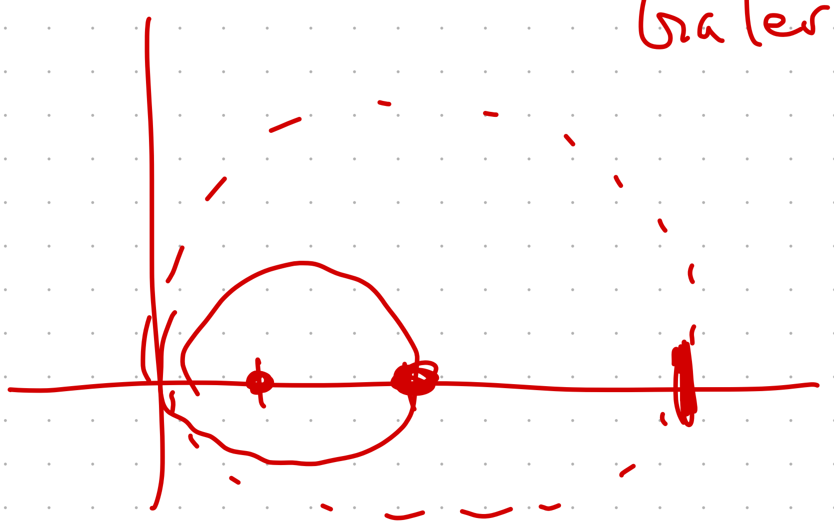
$\rightarrow$ A is symmetric.

② A is (semi) positive definite:

$\langle Lv, Lv \rangle = \|Lv\|^2 \geq 0$

③ Consider $L = u''$ (with b.c.)

Galerkin: (u', v') $\sim K(A) \sim O\left(\frac{1}{h^2}\right)$



$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ & & & -1 & 2 \end{bmatrix}$$

LS: $(u'', v'') \sim K(A) \approx O(h^{-4})$

$\hookrightarrow$ need higher order elements

Next up: a first-order system

Example

$$-u'' = f$$
$$u = 0 \quad \text{on } \Gamma$$

Let $q = u'$

Then
$$-q' = f$$
$$q - u' = 0$$

$$\begin{bmatrix} -\partial_x & 0 \\ I & -\partial_x \end{bmatrix} \begin{bmatrix} q \\ u \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

① $K \sim O(h^{-2})$

② introduce more variables !

Back to

$$-\nabla \cdot \underbrace{D \nabla u}_{\text{diff}} + \underbrace{b \cdot \nabla u}_{\text{convection}} + \underbrace{cu}_{\text{reaction}} = f$$

$$u = g_D \quad \text{on } \Gamma_D$$

$$n \cdot D \nabla u = g_N \quad \text{on } \Gamma_N$$

Three steps:

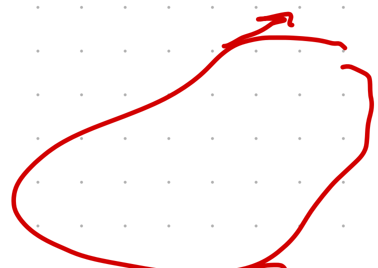
1. FOS-ize the problem
2. Identify L and the weak form
3. Show ellipticity (in something)

$$-\nabla\cdot D\nabla u + b\cdot\nabla u + cu = f$$

let $\underline{q} = D\nabla u$

$$\rightarrow \quad \begin{cases} (2) & -\nabla\cdot\underline{q} + b\cdot D^{-1}\underline{q} + cu = f \\ (1) & -\underline{q} + D\nabla u = 0 \end{cases}$$

$u = g_D$ on Γ_D $\longrightarrow$ $D^{-1}\underline{q}\cdot\underline{\tau} = 0$



$D^{-1}\underline{q}\times\underline{n} = 0$

$n\cdot(D\nabla u) = 0$ on Γ_N $\longrightarrow$ $n\cdot\underline{q} = 0$ on Γ_N

$$\underbrace{\begin{bmatrix} -I & D\nabla \\ -\nabla\cdot + b\cdot D^{-1} & cI \end{bmatrix}}_{L}\begin{bmatrix} \underline{q} \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

WHAT are we doing?

$$\min_{u,\underline{q}} \underbrace{\|D\nabla u-\underline{q}\|^2+\|-\nabla\cdot\underline{q}+\underline{b}\cdot D^{-1}\underline{q}+cu-f\|^2}_{G\left(\begin{bmatrix}\underline{q}\\u\end{bmatrix}\right)}$$

Make it easier: $D=1$

$\underline{b}=0$

$c=0$

(2) $L=\begin{bmatrix}-I & \nabla\\ -\nabla\cdot & 0\end{bmatrix}$

Find $\begin{bmatrix}\underline{q}\\u\end{bmatrix}$ s.t. $\left\langle \begin{bmatrix}-I & \nabla\\ -\nabla\cdot & 0\end{bmatrix}\begin{bmatrix}\underline{q}\\u\end{bmatrix}, \begin{bmatrix}-I & \nabla\\ -\nabla\cdot & 0\end{bmatrix}\begin{bmatrix}\underline{p}\\v\end{bmatrix}\right\rangle$

$$=\left\langle \begin{bmatrix}0\\f\end{bmatrix}, \begin{bmatrix}-I & \nabla\\ \nabla\cdot & 0\end{bmatrix}\begin{bmatrix}\underline{p}\\v\end{bmatrix}\right\rangle$$

③ is $\left\langle L\begin{bmatrix} q \\ u \end{bmatrix}, L\begin{bmatrix} p \\ v \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ f \end{bmatrix}, L\begin{bmatrix} p \\ v \end{bmatrix} \right\rangle$

elliptic?

Back to Galerkin

$a(u,v) = (u', v')$

or

$a(u,v) = (\nabla u, \nabla v)$

$\longrightarrow$ look at H^1.

$\|u\|_1^2 = \|u\|_0^2 + \|\nabla u\|_0^2$

Look at $\left\langle L\begin{bmatrix} q \\ u \end{bmatrix}, L\begin{bmatrix} p \\ v \end{bmatrix} \right\rangle$

$\downarrow$

$\left\langle L^* L\begin{bmatrix} q \\ u \end{bmatrix}, \begin{bmatrix} p \\ v \end{bmatrix} \right\rangle$

The (formal) adjoint

What is "$*$"?

Here it is the <u>formal</u> adjoint

For operator L, the <u>formal</u> adjoint L^* is the operator s.t.

$$\langle Lu, v\rangle_0 = \langle u, L^* v\rangle$$

for all (smooth) u, v with compact support.

Example

$$L = \partial_x$$

$$\langle Lu, v\rangle = \int_\Omega \partial_x u\, v = \cancel{uv\Big|_{\partial\Omega}} - \int_\Omega u\, \partial_x v$$

$$= \int u\,(-\partial_x)\, v$$

$$= \langle u, L^* v\rangle$$

$$L^* = -\partial_x$$

what about $\nabla^* = ?$ $\quad -\nabla\cdot$

$$\begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} \longrightarrow \begin{bmatrix} -\partial_x & -\partial_y \end{bmatrix}$$

$$\nabla \times^* = ?$$

$$= \nabla \times$$

Back to

$$\left\langle L^* L \begin{bmatrix} q \\ u \end{bmatrix}, \begin{bmatrix} p \\ v \end{bmatrix} \right\rangle$$

$L^* L$ is called the formal normal op.

→ a guide

$$L^* = \begin{bmatrix} -I & \nabla \\ -\nabla\cdot & 0 \end{bmatrix} \leftarrow L = \begin{bmatrix} -I & \nabla \\ -\nabla\cdot & 0 \end{bmatrix}$$

$$L^*L = \begin{bmatrix} I - \nabla\nabla\cdot & -\nabla \\ \nabla\cdot & -\nabla\cdot\nabla \end{bmatrix}$$

self-adjoint

$$\left\langle L^*L \begin{bmatrix} \underset{\sim}{q} \\ u \end{bmatrix}, \begin{bmatrix} \underset{\sim}{q} \\ u \end{bmatrix} \right\rangle$$

$$= \langle \underset{\sim}{q} - \nabla\nabla\cdot\underset{\sim}{q}, \underset{\sim}{q} \rangle + \langle -\nabla\cdot\nabla u, u \rangle$$

$$= \underbrace{\langle \underset{\sim}{q}, \underset{\sim}{q} \rangle + \langle \nabla\cdot\underset{\sim}{q}, \nabla\cdot\underset{\sim}{q} \rangle}_{H(\mathrm{div})} + \underbrace{\langle \nabla u, \nabla u \rangle}_{H^1}$$

$$\|\underset{\sim}{q}\|^2_{\mathrm{div}} \qquad |u|^2_1$$

$\rightarrow$ coercive / conf in

$$V = H_{div} \times H^1$$

$$\underset{\sim}{q} \qquad u$$

$\hat{u}, \hat{\underset{\sim}{q}}$ = exact

e = error

$$\| -\nabla(\hat{\underset{\sim}{q}} + e) \cdot - f \|^2 + \| -\hat{\underset{\sim}{q}} - \hat{e} + \nabla \hat{u} \|^2$$

$$= \underbrace{\| -\nabla \cdot e \|^2} + \| e \|^2$$

The fix:

$$L\left\{\begin{aligned} -\nabla\cdot \underline{q} + b\cdot D^{-1}\nabla u + cu &= f \\ -\underline{q} + D\nabla u &= 0 \\ \nabla\times D^{-1}\underline{q} &= 0 \end{aligned}\right.$$

<u>why?</u>

$$\nabla\times D^{-1}q = \nabla\times D^{-1}D\nabla u = \nabla\times\nabla u = 0$$

$$L^* L = \begin{bmatrix} I - \underline{\Delta} & -\nabla \\ \nabla\cdot & \underline{\Delta} \end{bmatrix}$$

$\rightarrow (H^1)^d \times H^1$ elliptic.