Today

- Stokes (cont, discrete)

Announcements

ICES

- HW5 due today
- Project 2 due a week from today
  - ↳ Need extra time due to conflict or other issues?
  - ↳ Contact me early.
- May do some work towards a second project topic for up to

→ 7% extra class credit.

- should only take ½ as much work as full-scale
- reduce scope in a suitable manner
- "critique" should remain
- please label which is full-scale and which is "extra"
- will post to Piazza with these rules

Example: Lagrange Multipliers in $\mathbb{R}^2$

$\to J(u) = \frac{1}{2} a(u,u) - (f,v)$

$$\begin{cases} f(x,y) = x^2 + y^2 & \to \quad \text{min!} \\ g(x,y) = x + y & = \quad 2 \end{cases}$$

Write down the Lagrangian.

$$\boxed{\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda g(x,y)}$$

Write down a necessary condition for a constrained minimum.

$$\boxed{0 = \nabla \mathcal{L} = \begin{pmatrix} \nabla f + \lambda \nabla g \\ g \end{pmatrix}}$$

Saddle Point Problems

X, M Hilbert spaces. $a : X \times X \to \mathbb{R}$ and $b : X \times M \to \mathbb{R}$ continuous bilinear forms, $f \in X'$, $g \in M'$. Minimize

$$J(u) = \frac{1}{2} a(u,u) - \langle f, u \rangle \qquad \text{subject to} \qquad b(u,\mu) = \langle g, \mu \rangle \quad (\mu \in M).$$

Apply the method of the Lagrange multipliers.

$$\mathcal{L}(u,\lambda) = J(u) + [\, b(u,\lambda) - (g,\lambda) \,]$$

Find $(u,\lambda) \in X \times M$

$$a(u,v) + b(v,\lambda) = (f,v) \quad (v \in X)$$

$$b(u,\mu) = (g,\mu) \quad (\mu \in M)$$

saddle point problem

Example: Saddle Point Problem in $\mathbb{R}^2$

$$
\begin{aligned}
f(x,y) = x^2 + y^2 \quad &\to \quad \text{min!} \\
g(x,y) = x + y \quad &= \quad 2
\end{aligned}
$$

Lagrangian: $\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda g(x,y) = x^2 + y^2 + \lambda(x + y - 2)$.

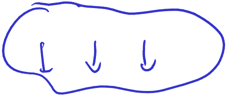
Show that $x = y = 1$, $\lambda = -2$ is a saddle point.

$$H_{\mathcal{L}} = \begin{pmatrix} H_f & \nabla g \\ \nabla g^T & 0 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ 0 & -B^T A^{-1} B \end{pmatrix}$$

Stokes Equation

$$
\begin{aligned}
\Delta \boldsymbol{u} + \nabla p &= -\boldsymbol{f} \quad (x \in \Omega),\\
\nabla \cdot \boldsymbol{u} &= 0 \quad (x \in \Omega),\\
\boldsymbol{u} &= \boldsymbol{u}_0 \quad (x \in \partial\Omega).
\end{aligned}
$$



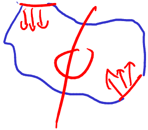
What are the pieces?

- $\vec{u}$ is velocity
- p is the pressure
- f is an externally applied force field

Stokes: Properties

$$\tilde{p} = p + c$$

$$\begin{aligned} \Delta \boldsymbol{u} + \nabla p &= -\boldsymbol{f} \quad (x \in \Omega), \\ \nabla \cdot \boldsymbol{u} &= 0 \quad (x \in \Omega), \\ \boldsymbol{u} &= \boldsymbol{u}_0 \quad (x \in \partial\Omega). \end{aligned}$$



Can we choose any $\boldsymbol{u}_0$?

$$\int_{\partial\Omega} \vec{u}_0 \cdot \vec{n} \, dS_x = \int_{\partial\Omega} \vec{u} \cdot \vec{n} = \int_{\Omega} \nabla \cdot \vec{u} = 0$$

Does Stokes fully determine the pressure?

→ only up to a const.

Stokes: Variational Formulation

$$\to \Delta \boldsymbol{u} + \nabla p = -\boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{u} = 0 \quad (x \in \partial\Omega).$$ $\Omega \subset \mathbb{R}^n$

Choose some function spaces (for homogeneous $\boldsymbol{u}_0 = 0$).

$X = H_0^1(\Omega) \times H_0^1(\Omega) = (H_0^1(\Omega))^2 \neq H^2$

$M = L_0^2(\Omega) = \{ q \in L^2 : \int_\Omega q \, dx = 0 \}$

Derive a weak form.

$a(\vec{u}, \vec{v}) = \int_\Omega Ju : Jv$ $\qquad b(\vec{v}, q) = \int (\nabla \cdot \vec{v}) q$

$A : B = tr(AB^T) = \sum_{ij} A_{ij} B_{ij}$

Find

$(-\operatorname{div})^* = \operatorname{grad}$

H_0^1: $\int \nabla \cdot \vec{u} \, v = \int \vec{u} \cdot \nabla v$

$a(\vec{u}, \vec{v}) + b(\vec{v}, p) = (\vec{f}, \vec{v}) \qquad \vec{v} \in X$

$b(\vec{u}, q) = 0 \qquad q \in M$

Solvability of Saddle Point Problems

The Stokes weak form is clearly in saddle-point form.
Do all saddle point problems have unique solutions?

$$f(x,y) = x^2 + y^2 \to \min$$

$$3 \cdot \left[\begin{aligned} x + y &= 2 \\ 3x + 3y &= 6 + \varepsilon \end{aligned}\right]$$

$$\mathcal{L}(x, y, \lambda, \bar{\lambda})$$

not uniquely defined

The inf-sup Condition

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ 0 & -B^T A^{-1} B \end{pmatrix}$$

$$\begin{aligned} a(u,v) + b(v,\lambda) &= \langle f, v\rangle \quad (v \in X), \\ b(u,\mu) &= \langle g, \mu\rangle \quad (\mu \in M). \end{aligned}$$

Theorem (Brezzi's splitting theorem (Braess, III.4.3))

The saddle point problem has a unique solution if and only if

- *The bilinear form $a(\cdot,\cdot)$ is V-elliptic, where $V = \{u : b(u,\mu) = 0 \text{for all } \mu \in M\}$, i.e. there exists $c_0 > 0$ so that*

$$a(v,v) \geq c_0 \|v\|_X^2 .$$

- *There exists a constant $c_2 > 0$ so that (inf-sup or LBB condition):*

$$\inf_{\mu \in M} \sup_{v \in X} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_M} \geq c_2 \iff \sup \frac{b(v,\mu)}{\|v\|_X} \geq c_2 \|\mu\|_M$$

inf-sup and Stokes

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) &= \displaystyle\int_\Omega J_{\boldsymbol{u}} : J_{\boldsymbol{v}}, \qquad \text{where } A : B = \operatorname{tr}(AB^T), \\ b(\boldsymbol{v}, q) &= \displaystyle\int_\Omega \nabla \cdot \boldsymbol{v} q. \end{cases}$$

Find $(\boldsymbol{u}, p) \in X \times M$ so that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{L^2} \quad (\boldsymbol{v} \in X), \\ b(\boldsymbol{u}, q) &= 0 \quad (q \in M). \end{aligned}$$

Theorem (Existence and Uniqueness for Stokes (Braess, III.6.5))

There exists a unique solution of this system when $\boldsymbol{f} \in H^{-1}(\Omega)^n$.

(based on results due to Ladyšenskaya, Nečas)

Discretizations for Stokes

Demo: 2D Stokes Using Firedrake (P^1-P^1)

Give a heuristic reason why P^1-P^1 might not be great.

Diff. operators on u and p are of diff. order.

Demo: Bad Discretizations for 2D Stokes

Establishing a Discrete inf-sup Condition

Suppose $b : X \times M \to \mathbb{R}$ satisfies inf-sup. Subspaces $X_h \subseteq X$, $M_h \subseteq M$.

Fortin's Criterion ([Fortin 1977])

Suppose there exists a bounded projector $\Pi_h : X \to X_h$ so that

$$b(v, \mu_h) = b(\Pi_h v, \mu_h) \quad (v \in X) \iff b(v - \Pi_h v, \mu_h) = 0 \quad (v \in X, \mu_h \in M_h)$$

If $\|\Pi_h\| \le c$ for some constant c independent of h, then b satisfies the inf-sup condition on $X_h \times M_h$.

Let $\mu_h \in M_h$

$$\sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|} \ge \sup_{v_h \in \Pi_h X} \frac{b(v_h, \mu_h)}{\|v_h\|} = \sup_{v \in V} \frac{b(\Pi_h v, \mu_h)}{\|\Pi_h v\|} = \sup \frac{b(v, \mu_h)}{\|\Pi_h v\|}$$

$$\ge \frac{1}{c} \sup \frac{b(v, \mu_h)}{\|v\|} \ge c_2 \|\mu_h\|.$$

H^1-Boundedness of the L^2-Projector

Assume H^2-regularity and a uniform triangulations $\mathcal{T}_h$. (Not in general!)

H^1-Boundedness of the L^2-Projector (Braess Corollary II.7.8)

Let π_h^0 be the L_2-projector onto a finite element space $V_h \subset H^1(\Omega)$. Then, for an h-independent constant c,

$$\|\pi_h^0 v\|_{H^1} \le \|v\|_{H^1}$$

Ingredients?

- Ell. reg.
- Aubin-Nitsche
- Inv. est.

$v_h \in V_h$ $\quad \cancel{v \in V}$

$$\|v_h\|_{H^\ell} \le C h^{m-\ell} \|v_h\|_{H^m}$$

$$\|v_h\|_{H^1} \le C h^{-1} \|v_h\|_{L^2}$$

H^1-Boundedness of the L^2-Projector

Does H^1 boundedness of the H^1 projector hold?

How would this break down without the uniformity assumption?

Bubbles and the MINI Element

What is a bubble function?

$$\varphi_b(r,s) = r \cdot s(1-r-s)$$

Let B^3 be the span of the bubble function and $\mathcal{T}_h$ the triangulation.

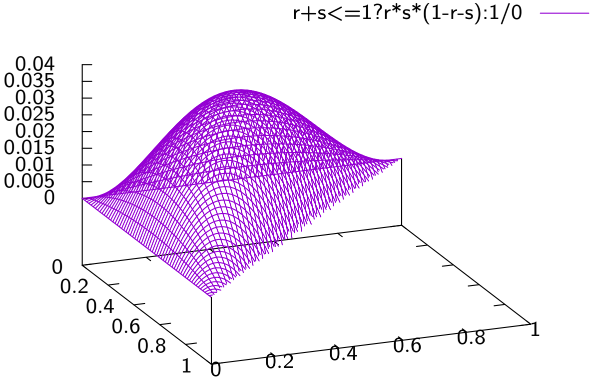
Define the MINI variational space $X_h \times M_h$.

$$X_h = P^1 + B^3$$

$$M_h = P^1$$

Computational impact of the bubble DOF?

The Bubble in Pictures



MINI Satisifies an inf-sup Condition (1/4)

MINI satisifes inf-sup (Braess Theorem III.7.2)

Assume Ω is convex or has a smooth boundary. Then the MINI variational space satisfies an inf-sup condition for every variational form that itself satisfies one.

MINI Satisifies an inf-sup Condition (2/4)

Create a projector onto the bubble space B^3.

$\pi_h^{1\cdot} : L^2 \to B^3$

$\int_E (\pi_h^1 v - v) \stackrel{!}{=} 0 \quad \text{for } E \in \mathcal{T}_h$

What does this bubble projector do?

- proj. const.
- replace with $\int =$ const.

Do we have an estimate for the bubble projector?

MINI Satisifies an inf-sup Condition (3/4)

Make an overall projector Π_h onto X_h.

Show Fortin's criterion for Π_h.

MINI Satisifies an inf-sup Condition (4/4)

- $\left\|\pi_h^0 v\right\|_{H^1} \leq c_1 \left\|v\right\|_{H^1}$ for L^2 projector $\pi_h^0 : H_0^1 \to \mathcal{M}_h$.
- $\left\|v - \pi_h^0 v\right\|_{L^2} \leq c_2 h \left|v\right|_{H^1}$.
- $\left\|\pi_h^1 v\right\|_{L^2} \leq c_3 \left\|v\right\|_{L^2}$.

Show H^1-boundedness of Π_h.

Demo

Demo: 2D Stokes Using Firedrake (MINI and Taylor-Hood)