1D Advection Equation and Characteristics

a<0
↳ char. curves move left

$$u_t + au_x = 0, \quad u(0,x) = g(x) \qquad (x \in \mathbb{R})$$

a>0
↳ char. curves slant right

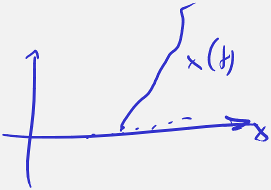
Solution?

$f(u) = a u$

$u_t + f(u)_x = 0$

$$\frac{dx(t)}{dt} = f'(u(x,t))$$

$x(0) = x_0$



$$\frac{du(x(t),t)}{dt} = \dots = 0$$

$u_{tt} = c^2 u_{xx}$

Solving Advection with Characteristics

$$u_t + au_x = 0, \quad u(0,x) = g(x) \qquad (x \in \mathbb{R})$$

Find the characteristic curve for advection.

$x(t) = x_0 + at$

Generalize this to a solution formula.

$u(x,t) = g(x - at)$ $\qquad f'(u) = a =$ adv. speed

Does the solution formula admit solutions that aren't obviously allowed by the PDE?

sol. formula admits non-smooth solutions

↳ e.g. solutions having jumps

→ later: generalize / weaken notion of "derivative"

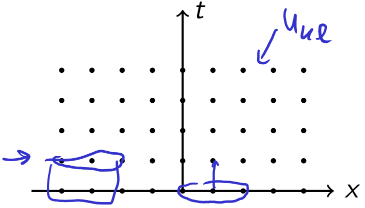
Finite Difference for Hyperbolic: Idea

$$\{(x_k, t_\ell) : x_k = kh_x, t_\ell = \ell h_t\}$$

If $u(x,t)$ is the exact solution, want

$u_t + a u_x = 0$

$$u_{k,\ell} \approx u(x_k, t_\ell).$$



Condition at each grid point?

- choose a derivative stencil for each occurring derivative
- plug into PDE
- get giant system of eqns.
- solve

What are explicit/implicit schemes?

assumption: step fwd. in time
- next time level found by
- system of eqns (impl.) / formula (expl.)

Designing Stencils

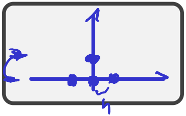
$u_t + a u_x = 0$

$u_{k,\ell}$

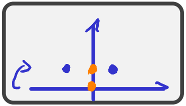
$\ell \to$ time

$k \to$ space

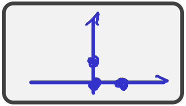
ETCS:



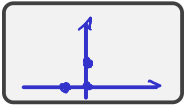
ITCS:



ETFS:



ETBS:



Terminology?

- E/I: Explicit / Implicit
- T/S: Time/Space
- F/B/C: Forward (right) / Backward (left) / Centered
- Upwind / Downwind

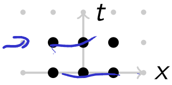
  $a>0$: upwind = backward

  downwind = forward

Write out ITCS:

$$\frac{u_{k,\ell+1}-u_{k,\ell}}{h_t} + a\,\frac{u_{k+1,\ell+1}-u_{k-1,\ell+1}}{2h_x} = 0$$

Crank-Nicolson



Crank-Nicolson

Write out Crank-Nicolson:

$$\boxed{\frac{u_{k,\ell+1}-u_{k,\ell}}{h_t} + \frac{a}{2}\left[\frac{u_{k+1,\ell}-u_{k-1,\ell}}{2h_x} + \frac{u_{k+1,\ell+1}-u_{k-1,\ell+1}}{2h_x}\right] = 0}$$

Would like: two-level schemes only.

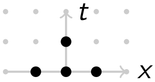
Lax-Wendroff

$u_t + a u_x = 0$

$\to u_t = -a u_x$

What's the core idea behind Lax-Wendroff?

- Write a Taylor expn. in time
- Use the PDE to replace ∂_t with ∂_x
- Discretize space ∂_x as before



Lax-Wendroff

Write out Lax-Wendroff.

$u_t = -a u_x \quad \Big| \quad u_{tt} = (-a u_x)_t = -a (u_t)_x = a^2 u_{xx}$

$$u_{k,\ell+1} - u_{k,\ell} \approx h_t u_t(x_k, t_\ell) + \frac{h_t^2}{2} u_{tt}(x_k, t_\ell)$$

$$= -h_t a u_x(x_k, t_\ell) + \frac{a^2 h_t^2}{2} u_{xx}(x_k, t_\ell)$$

$$\approx -h_t a \frac{u_{k+1,\ell} - u_{k-1,\ell}}{2h_x} + \frac{a^2 h_t^2}{2} \frac{u_{k+1,\ell} - 2u_{k,\ell} + u_{k-1,\ell}}{h_x^2}.$$

Exploring Advection Schemes

Demo: Methods for 1D Advection [cleared]

- Which of the schemes "work"?
- Any restrictions worth noting?

Outline

A Matrix View of Two-Level Stencil Schemes

Define

$$\vec{v}_\ell = \boldsymbol{v}_\ell = \begin{bmatrix} u_{1,\ell} \\ \vdots \\ u_{N_x,\ell} \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_{N_t} \end{bmatrix}.$$

Define

$$\boldsymbol{u}_\ell = \begin{bmatrix} u(x_1, t_\ell) \\ \vdots \\ u(x_{N_x}, t_\ell) \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_{N_t} \end{bmatrix}.$$

Definition (Two-Level Finite Difference Scheme)

A finite difference scheme that can be written as

$$P_h \vec{v}_{\ell+1} = Q_h \vec{v}_\ell + h_t \vec{b}_\ell$$

is called a two-level linear finite difference scheme.

- Mostly will (tacitly) assume $\vec{b}_\ell = \vec{0}$. (i.e. no src term)
- P_h, Q_h may depend on h_x, h_t
- might also consider infinite (spatial) vectors.

Rewriting Schemes in Matrix Form (1/2)

$$P_h \boldsymbol{v}_{\ell+1} = Q_h \boldsymbol{v}_\ell + h_t \boldsymbol{b}_\ell$$

Find P_h and Q_h for ETCS: