A Matrix View of Two-Level Stencil Schemes

- hw1 now due Feb 11
- hw2 out before Wed
- quiz?

Numerical solution vectors:

$$\vec{v}_\ell = \boldsymbol{v}_\ell = \begin{bmatrix} u_{1,\ell} \\ \vdots \\ u_{N_x,\ell} \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_{N_t} \end{bmatrix}.$$

True solution vectors:

$$\boldsymbol{u}_\ell = \begin{bmatrix} u(x_1, t_\ell) \\ \vdots \\ u(x_{N_x}, t_\ell) \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_{N_t} \end{bmatrix}.$$

Definition (Two-Level Finite Difference Scheme)

A finite difference scheme that can be written as

$$\boxed{P_h \vec{v}_{\ell+1} = Q_h \vec{v}_\ell + h_t \vec{b}_\ell}$$

is called a two-level linear finite difference scheme.

- Mostly $\vec{b}_\ell = \vec{0}$
- $P_h(h_x, h_t)$ $Q_h(h_x, h_t)$

Rewriting Schemes in Matrix Form (1/2)

$$P_h \boldsymbol{v}_{\ell+1} = Q_h \boldsymbol{v}_\ell + h_t \boldsymbol{b}_\ell$$

Find P_h and Q_h for ETCS:

ETCS: $\dfrac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\dfrac{u_{k+1,\ell} - u_{k-1,\ell}}{2h_x} = 0$

$$u_{k,\ell+1} = u_{k,\ell} + \frac{ah_t}{2h_x}\left(u_{k+1,\ell} - u_{k-1,\ell}\right)$$

$P_h = I$ / $Q_h = \text{tridiag}\left(\dfrac{ah_t}{2h_x}, 1, -\dfrac{ah_t}{2h_x}\right)$

Rewriting Schemes in Matrix Form (2/2)

Find P_h and Q_h for Crank-Nicolson:

$$P_h = \text{tridiag}\left(-\frac{ah_t}{4h_x},\ 1,\ \frac{ah_t}{4h_x}\right)$$

$$Q_h = \text{tridiag}\left(\frac{ah_t}{4h_x},\ 1,\ -\frac{ah_t}{4h_x}\right)$$

Truncation Error

Definition (Truncation Error)

The local truncation error $\tau_{k,\ell}$ is the error that remains when the FD method is applied to an exact sol.

Demo: Truncation Error Analysis via sympy [cleared]

Error and Error Propagation

$P_h \vec{v}_{\ell+1} = Q_h \vec{v}_\ell$

Express definition of truncation error in our two-level framework:

$$P_h \vec{u}_{\ell+1} = Q_h \vec{u}_\ell + \underbrace{\vec{\tau}_\ell}_{\text{trunc. error}} h_t$$

Define $\boldsymbol{e}_\ell = \boldsymbol{u}_\ell - \boldsymbol{v}_\ell$. Understand the error as accumulation of truncation error:

$$\vec{e}_0 = \vec{0}$$

$$P_h \vec{u}_{\ell} = Q \vec{u}_\ell + \vec{\tau}_\ell h_t$$

$$- \quad P_h \vec{v}_{\ell+1} = Q_h \vec{v}_\ell$$

$$P_h \vec{e}_{\ell+1} = Q_h \vec{e}_\ell + \vec{\tau}_\ell h_t$$

$$\vec{e}_{\ell+1} = P_h^{-1} Q_h \vec{e}_\ell + P_h^{-1} \vec{\tau}_\ell h_t$$

Discrete and Continuous Norms

To measure properties of numerical solutions we need norms. Define a discrete L^∞ norm.

$$\|\vec{e}\|_\infty = \max_{k,\ell} |e_{k,\ell}|$$

Define a discrete L^2 norm.

$v \in \mathbb{R}^n$

$$\|\vec{e}\|_2 = \sqrt{\sum_{k,\ell} h_x h_t e_{k,\ell}^2}$$

$$\|\vec{e}\|$$

$$\underbrace{\|v\|_2 = \sqrt{\sum_{i=1}^n v_i^2}}_{\ell^2}$$

Important features:

$$\|f\| = \sqrt{\int f^2}$$

– $\|e\|_{2/\infty}$ should not change wildly as we consider limits $h_x, h_t \to 0$

Consistency and Convergence

Assume $u, (\partial_x^{q_x})u, (\partial_t^{q_t})u \in L^2(\mathbb{R} \times [0, t^*])$.

Definition (Consistency)

A two-level scheme is consistent in the L^2-norm with order q_t in time and q_x in space if

$$\max_{\ell,\, \ell h_t \le t^*} \|\vec{\tau}_\ell\| = O(h_x^{q_x} + h_t^{q_t}) \quad (\text{as } h_x, h_t \to 0)$$

Definition (Convergence)

A two-level scheme is convergent in the L^2-norm with order q_t in time and q_x in space if

$$\max_{\ell,\, \ell h_t \le t^*} \|\vec{e}_\ell\| = O(h_x^{q_x} + h_t^{q_t}) \quad (\text{as } h_x, h_t \to 0)$$

Analyzing ETFS (1/2)

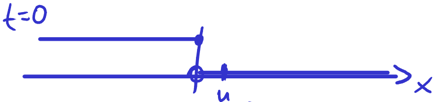
$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\frac{u_{k+1,\ell} - u_{k,\ell}}{h_x} = 0$$

Let's understand more precisely what happens for this scheme. Assume $a > 0$.

Rewrite

$$u_{k,\ell+1} = u_{k,\ell} - \frac{\overbrace{a\,h_t}^{\lambda}}{h_x}\left(u_{k+1,\ell} - u_{k,\ell}\right) = (1+\lambda)\,u_{k,\ell} - \lambda\,u_{k+1,\ell}$$

Analyzing ETFS (2/2)



$$u_{k,\ell+1} = (1+\lambda)u_{k,\ell} - \lambda u_{k+1,\ell}$$

Consider $u(x,0) = 1_{[-1,0]}(x)$. Predict solution behavior.

$u_{0,0} = 1 \qquad u_{1,\ldots,0} = 0$

$u_{0,1} = 1+\lambda \qquad u_{1,\ldots,1} = 0$

$u_{0,2} = (1+\lambda)^2 \qquad u_{1,\ldots,2} = 0$

$\vdots \qquad \vdots$

$\left(1+\frac{x}{n}\right)^n \to e^x$

$$u(0,t) \approx u_{0,t/h_t} = \left(1+\frac{a h_t}{h_x}\right)^{t/h_t} = \left(1+\frac{at/h_x}{t/h_t}\right)^{t/h_t} \to \exp\left(\frac{at}{h_x}\right)$$

Demo: Methods for 1D Advection [cleared] (Revisit ETFS)

Stability

$$P_h \boldsymbol{v}_{\ell+1} = Q_h \boldsymbol{v}_\ell$$

Write down a matrix product to bring $\boldsymbol{v}_0$ to $\boldsymbol{v}_\ell$:

$$\vec{v}_\ell = (P_h^{-1} Q_h)^\ell \vec{v}_0$$

Definition (Stability)

A two-level scheme is stable in the L^2-norm if there exists a constant $c > 0$ independent of h_t and h_x so that

$$\left\| (P_h^{-1} Q_h)^\ell P_h^{-1} \right\| \leq c$$

for all ℓ and h_t such that $\ell h_t \leq t^*$.

Lax Convergence Theorem

Theorem (Lax Convergence)

If a two-level FD scheme is

- *consistent in the L^2-norm with order q_t in time and q_x in space, and*
- *stable in the L^2-norm, then*

it is convergent in the L^2-norm with order q_t in time and q_x in space.

- Stronger result holds: "if and only if"
  Lax equiv. thm. / Lax-Richtmyer thm.
- "Fundamental thm of numerical analysis"
  consistency + stability $\Rightarrow$ convergence
- A related result holds for ODEs
  (Dahlquist)

Lax Convergence: Proof (1/2)

Lax Convergence: Proof (2/2)

$$\|\boldsymbol{e}_\ell\| = h_t \sum_{m=1}^{\ell} \left\|(P_h^{-1} Q_h)^{\ell-m} P_h^{-1}\right\| \boldsymbol{\tau}_{m-1}.$$

Conditions for Stability

$$\left\| (P_h^{-1} Q_h)^\ell P_h^{-1} \right\| \leq c$$

Give a simpler, sufficient condition:

How can we show bounds on these matrix norms?