HW 1 due Friday

Consistency and Convergence

Assume $u, (\partial_x^{q_x})u, (\partial_t^{q_t})u \in L^2(\mathbb{R} \times [0, t^*])$.

Definition (Consistency)

A two-level scheme is consistent in the L^2-norm with order q_t in time and q_x in space if

$$\max_{l,\ l h_t \le t^*} \|\vec{t}_l\| = O(h_x^{q_x} + h_t^{q_t}) \quad \text{as} \quad \begin{matrix} h_x \to 0 \\ h_t \to 0 \end{matrix}$$

Definition (Convergence)

A two-level scheme is convergent in the L^2-norm with order q_t in time and q_x in space if

$\vec{u}_l - \vec{v}_l$

$$\max_{l,\ l h_t \le t^*} \|\vec{e}_l\| = O(h_x^{q_x} + h_t^{q_t}) \quad \text{as} \quad \begin{matrix} h_x \to 0 \\ h_t \to 0 \end{matrix}$$

Stability

$$P_h \boldsymbol{v}_{\ell+1} = Q_h \boldsymbol{v}_\ell$$

Write down a matrix product to bring $\boldsymbol{v}_0$ to $\boldsymbol{v}_\ell$:

$$\vec{v}_\ell = (P_h^{-1} Q_h)^\ell \vec{v}_0$$

Definition (Stability)

A two-level scheme is stable in the L^2-norm if there exists a constant $c > 0$ independent of h_t and h_x so that

$$\left\| (P_h^{-1} Q_h)^\ell P_h^{-1} \right\| \leq c$$

for all ℓ and h_t such that $\ell h_t \leq t^*$.

Lax Convergence Theorem

Theorem (Lax Convergence)

If a two-level FD scheme is

- *consistent in the L^2-norm with order q_t in time and q_x in space, and*
- *stable in the L^2-norm, then*

it is convergent in the L^2-norm with order q_t in time and q_x in space.

Lax Convergence: Proof (1/2)

$$P_h \vec{e}_{\ell+1} = \underbrace{Q_h \vec{e}_\ell}_{\text{propagated}} + \underbrace{\vec{\tau}_\ell h_t}_{\text{trunc. error}}$$

$$\vec{e}_{\ell+1} = P_h^{-1} Q_h \vec{e}_\ell + P_h^{-1} \vec{\tau}_\ell h_t$$

Recall. $\vec{e}_0 = 0$. (assumption)

$$\vec{e}_1 = h_t P_h^{-1} \vec{\tau}_0$$

$$\vec{e}_2 = h_t (P_h^{-1} Q_h) P_h^{-1} \tau_0 + P_h \vec{\tau}_1 h_t$$

By induction:

$$\vec{e}_\ell = h_t \sum_{m=1}^{\ell} \underbrace{(P_h^{-1} Q_h)^{\ell-m} P_h^{-1}} \vec{\tau}_{m-1}$$

Lax Convergence: Proof (2/2)

$\|Ax\| \leq \|A\| \, \|x\|$

$$\boldsymbol{e}_\ell = h_t \sum_{m=1}^{\ell} (P_h^{-1} Q_h)^{\ell-m} P_h^{-1} \boldsymbol{\tau}_{m-1}.$$

$0 \leq \ell h_t \leq t^*$

$$\|e_\ell\| \leq h_t \sum_{m=1}^{\ell} \|(P_h^{-1}Q_h)^{\ell-m} P_h^{-1} \vec{\tau}_{m-1}\|$$

$$\leq h_t \sum_{m=1}^{\ell} \underbrace{\|(P_h^{-1}Q_h)^{\ell-m} P_h^{-1}\|}_{\leq c \text{ (stab)}} \, \|\vec{\tau}_{m-1}\|$$

$$\leq \underbrace{h_t \ell}_{\leq t^*} \, c \, \underbrace{\max_{\ell h_t \leq t^*} \|\tau_\ell\|}_{= O(h_x^{q_x} + h_t^{q_t})} = t^* c \, O(h_x^{q_x} + h_t^{q_t})$$

consistency

Conditions for Stability

$$\left\|(P_h^{-1}Q_h)^\ell P_h^{-1}\right\| \le c$$

Give a simpler, sufficient condition:

$$\|P_h^{-1}Q_h\| \le 1 \qquad \|P_h^{-1}\| \le c$$

↳ Lax-Richtmyer stability

How can we show bounds on these matrix norms?

- bounds have to hold for all h_x, h_t
- proving this: generally cumbersome
- to prove: bound singular values

Stability of ETBS (1/3)

Theorem (Gershgorin)

For a matrix $A \in \mathbb{C}^{N\times N} = (a_{i,j})$,

$$\sigma(A) \subset \bigcup_{j=1}^{N} \bar{B}\left(a_{j,j}, \sum_{k\neq j} |a_{j,k}|\right).$$

"the spectrum"
(all eigenvalues)

ETBS:

$$\frac{u_{k,\ell+1} - u_{k,l}}{h_t} + a\frac{u_{k,\ell} - u_{k-1,\ell}}{h_x} = 0$$

Analyze stability of ETBS:

Let $\lambda = \frac{a h_t}{h_x}$. $u_{k,\ell+1} = \lambda u_{k-1,\ell} + (1-\lambda) u_{k,\ell}$.

$P_h = I$ $\quad Q_h = \mathrm{tridiag}(\lambda, 1-\lambda, 0)$. $\quad \|P_h^{-1}\| \leq 1$.

Stability of ETBS (2/3)

$P_h = I$ and $Q_h = \text{tridiag}(\lambda, 1-\lambda, 0)$.

Consider singular values of $P_h^{-1}Q_h = Q_h$: eigenvalues of $Q_h^TQ_h$

$Q_h^TQ_h = \text{tridiag}(\lambda(1-\lambda), (1-\lambda)^2+\lambda^2, \lambda(1-\lambda))$. Assume: $0 \le \lambda \le 1$

$\Rightarrow \lambda(1-\lambda) \ge 0$. Let Λ be an eigenvalue of $Q_h^TQ_h$.

$$2\lambda^2 - 2\lambda \le \Lambda - (1-\lambda)^2 - \lambda^2 \le \underbrace{2\lambda - 2\lambda^2}_{\ge 0}$$

$$\Lambda - 1 + 2\lambda - 2\lambda^2$$

$$1 - 4\lambda + 4\lambda^2 \le \Lambda \le 1$$

$$0 \le (1-2\lambda)^2 = \Lambda \le 1$$

"Analogously", if $\lambda \notin [0,1]$, $|\Lambda|$ is bounded below by 1.

Stability of ETBS (3/3)

Summarize ETBS stability:

ETBS is stable if and only if $0 \leq \lambda \leq 1$.

"conditional stability"

$$0 \leq \frac{a h_t}{h_x} \leq 1 \iff h_t \leq \frac{h_x}{a}$$

Courant-Friedrichs-Lewy condition "CFL" condition

Comments?