# Studying Solutions of the PDE

Saw numerically: interesting dispersion/dissipation behavior.
Want: theoretical understanding.

Consider *linear*, *continuous* (not yet discrete) differential operators

$$
\begin{aligned}
L_1 u &= u_t + a u_x, \\
L_2 u &= u_t - D u_{xx} + a u_x \qquad (D > 0) \\
L_3 u &= u_t + a u_x - \mu u_{xxx}.
\end{aligned}
$$

What could we use as a prototype solution?

# A Prototype Solution of the PDE

$e^{i\varphi} = \cos\varphi + i\sin\varphi$

$u(x,t) = f(x - ct)$

Observation: all these operators are diagonalized by complex exponentials.
Come up with a 'prototype complex exponential solution'.

$$z(x,t) = z_0 e^{i(\underbrace{kx - \omega t})}$$

What type of function is this?

$k, \omega \in \mathbb{R}$: traveling wave w/ speed $c = \frac{\omega}{k}$

$k$ imaginary: spatially decaying, "damped" wave

$\operatorname{Im} \omega < 0$: wave decaying in time

# Wave-like Solutions of the PDE

$L_2 u = u_t - D u_{xx} + a u_x$

$L_2 z = z\,(\underbrace{-i\omega + Dk^2 - aik})$

$$z(x,t) = z_0 e^{i(kx - \omega t)}$$

Observations in connection with $L$?

- $Lz = \lambda(\omega, k)\, z$
- $Lz = 0 \iff \lambda(\omega, k) = 0$

What is the dispersion relation?

$\lambda(\omega, k) = 0$ is the d.r. for the PDE $L$.

# Picking Apart the Dispersion Relation

$\alpha \in \mathbb{R}, \beta \in \mathbb{R}$

Consider $\omega(k) = \alpha(k) + i\beta(k)$. Rewrite the wave solution with this.

dispersion

dissipation

$$z(x,t) = z_0 e^{i(kx - \omega(k)t)}$$
$$= z_0 e^{i(kx - \alpha(k)t - i\beta(k)t)}$$
$$= z_0 e^{\beta k(t)} \underbrace{e^{i(kx - \alpha(k)t)}}_{|\cdot| = 1}$$

How can we recognize dissipation?

If $\beta(k) < 0$, we call the PDE dissipative

What is the phase speed? How can we recognize dispersion?

- The phase speed is $v_{ph}(k) = \frac{\alpha(k)}{k}$.
- If ($v_{ph}(k) \equiv$ const. $\Leftrightarrow$ $\alpha(k)$ is linear in $k$), all waves move at the same speed
- If that's not the case, we will call the PDE dispersive.

## Dispersion Relation: Examples

In each case, find the dispersion relation and identify properties.

$L_1 u = u_t + a u_x$

$\lambda(\omega,k) = i(ak-\omega) = 0 \iff \omega = ak$

no dissipation, no dispersion

$L_2 u = u_t - D u_{xx} + a u_x$ $(D > 0)$

$e^{i(kx-\omega t)}$

$\lambda(\omega,k) = -i\omega + Dk^2 + iak = 0 \iff \omega = ak - iDk^2$

no dispersion, yes dissipation

$L_3 u = u_t + a u_x - \mu u_{xxx}$

$\lambda(\omega,k) = -i\omega + iak + i\mu k^3 = 0 \iff \omega = ak + \mu k^3$

no dissipation, yes dispersion

# Numerical Dissipation/Dispersion Analysis

Goal: Want discrete finite difference scheme to match dissipation/dispersion behavior of continuous PDE.

Define a discrete wave-like function:

$$z_{j,\ell} = e^{i(k\,jh_x \quad - \omega\,\ell h_t)}$$

We want $\boldsymbol{z}$ to solve $P_h \boldsymbol{z}_{\ell+1} = Q_h \boldsymbol{z}_\ell$. How can we connect the operators to the wave solution?

Assume $P_h$ and $Q_h$ are Toeplitz.

# Toeplitz and Waves

$$z_{j,\ell} = z_0 e^{i(kjh_x - \omega \ell h_t)}.$$

?

**Theorem (Waves Diagonalize Toeplitz Operators)**

*Let $T$ be a Toeplitz operator. Then $T\boldsymbol{z}_\ell = \lambda(k)\boldsymbol{z}_\ell = \hat{\boldsymbol{t}}(kh_x)\boldsymbol{z}_\ell$.*

$$(T z_\ell)_j = \sum_m z_{m,\ell}\, t_{j-m} = \sum_m z_0 e^{i(kmh_x - \omega \ell h_t)}\, t_{j-m}$$

$$= \left(\sum_m z_0 e^{i(k(m-j)h_x)}\, t_{j-m}\right) e^{i(kjh_x - \omega \ell h_t)}$$

$m' = j - m$

$$= \left(\sum_{m'} e^{-ikm'h_x}\, t_{m'}\right) z_{j,\ell}$$

$$= \hat{t}(kh_x)\, z_{j,\ell} \qquad \Rightarrow \lambda(k) = \hat{t}(kh_x)$$

# Waves and Two-Level Schemes

Since $P_h$ and $Q_h$ are Toeplitz, we must have

$$\underbrace{P_h \mathbf{z}_{\ell+1} = \lambda_P(k)}_{\hookrightarrow kh_x =: \kappa} \mathbf{z}_{\ell+1}, \qquad Q_h \mathbf{z}_\ell = \underline{\lambda_Q(k)} \mathbf{z}_\ell.$$

What does that mean?

$$\lambda_p(\kappa)\, \vec{z}_{\ell+1} = \lambda_Q(\kappa)\, \vec{z}_\ell$$

$$\lambda_p(\kappa)\, e^{-i\omega h_t}\, \cancel{\vec{z}_\ell} = \lambda_Q(\kappa)\, \cancel{\vec{z}_\ell}$$

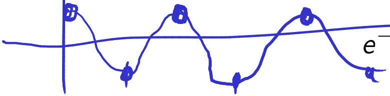
$$e^{-i\omega h_t} = \frac{\lambda_Q(\kappa)}{\lambda_p(\kappa)} = \frac{\dot{q}(\kappa)}{\dot{p}(\kappa)} = s(\kappa) = s(kh_x)$$

Seen before?

in vN stability

# Discrete Dispersion Relation (1/2)

So $\boldsymbol{z}_\ell$ is a solution of the finite difference scheme if $\omega = \omega(kh_x)$ satisfies



$$e^{-i\omega(\kappa)h_t} = s(\kappa),$$

$PPW = \frac{1}{\kappa}$

2 PPW

where we let $\kappa = kh_x$. Interpret $\kappa$.

the number of wavelengths per point

Let $s(\kappa) = |s(\kappa)|\, e^{i\varphi(\kappa)} = e^{\log|s(\kappa)| + i\varphi(\kappa)}$. $\omega(\kappa)$?

mag. ⇒ polar angles

$$\omega = \frac{-\varphi(\kappa) + i \log|s(\kappa)|}{h_t}$$

# Discrete Dispersion Relation (2/2)

$$\omega(\kappa) = \frac{-\varphi(\kappa) + i \log |s(\kappa)|}{h_t}.$$

Plug that into the wave-like solution:

$$z_{j,\ell} = z_0 e^{i(\kappa j h_x - \omega(\kappa) \ell h_t)}$$

$$= z_0 \exp\left(i\left(\kappa j h_x - \frac{-\varphi(\kappa) + i \log|s(\kappa)|}{h_t} \ell h_t\right)\right)$$

$$= z_0 \underbrace{\exp(\log(|s(\kappa)|)\ell)}_{|s(\kappa)|^\ell} \exp\left(i\kappa\left(j h_x - \frac{-\varphi(\kappa)}{\kappa h_t} \ell h_t\right)\right)$$

Criterion for stability?

$|s(\kappa)| \le 1$ (as with vN)

# Numerical Dispersion/Dissipation

Finite difference scheme $P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell$ with symbol $s(k)$.

$$z_{j,\ell} = \underbrace{z_0 e^{\log|s(\kappa)|\ell} e}_{} {}^{ik\left(jh_x - \frac{-\varphi(\kappa)}{kh_t}\ell h_t\right)}$$

When is the scheme dissipative?

if $|s(kh_x)| < 1$, then the scheme is dissipative.
" with factor $|s(kh_x)|$".

What is the phase speed?

$$v_{ph} = \frac{-\varphi(kh_x)}{kh_t}$$

Dispersion?

If $v_{ph} \equiv$ const, the scheme is non-dispersive.
Otherwise, it is.

# Dispersion/Dissipation Analysis of ETBS

Let $\lambda = ah_t/h_x$. Shown earlier: $s(kh_x) = 1 - \lambda(1 - e^{-ikh_x})$.