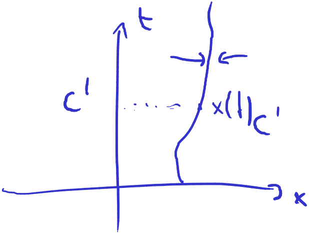
HW:
- Analyze Leapfrog
- FD + BCs: stability
- FD + Burgers

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad \Longleftrightarrow \qquad u_t + u\,u_x = 0$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

- Feedback



$$\underbrace{\frac{d}{dt}\Bigg(\underbrace{\int_a^{x(t)} u\,dx}_{G_a}}_{} + \underbrace{\int_{x(t)}^{b} u\,dx}_{G_b} \Bigg) + f(u(b,t)) - f(u(a,t)) = 0$$

$$G_a(z,t) = \int_a^z u(x,t)\,dx$$

$$\frac{d}{dt}\,G_a(x(t),t)$$

Rankine-Hugoniot Condition (2/2)

$$(d/dt)G_a(x(t),t) = u(x(t),t)x'(t) - (f(u(x(t),t)) - f(u(a,t))).$$

$u^- := \lim_{\xi \to x(t)^-} u(\xi, t)$ $\qquad$ $u^+ := u(x(t)^+, t)$

$$d\, G_a(x(t),t) = u^- x'(t) - (f(u^-) - f(u(a,t)))$$

$$d\, G_b(x(t),t) = -u^+ x'(t) - (f(u(b,t)) - f(u^+))$$

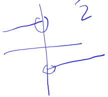
$$u^- x'(t) - f(u^-) - u^+ x'(t) + f(u^+) = 0$$

$$\boxed{x'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-}}$$

R-H condition

$\frac{u^2}{2}$

$\frac{1}{2} - \frac{1}{2}$



Rankine-Hugoniot and Weak Solutions

Theorem (Rankine-Hugoniot and Weak Solutions)

If u is piecewise C^1 and is discontinuous only along isoated curves, and if u satisfies the PDE when it is C^1, and the Rankine-Hugoniot condition holds along all discontinuous curves, then u is a weak solution of the conservation law.

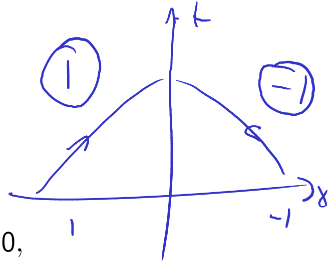
$$u_t + \nabla \cdot \vec{f}(u) = 0$$

Riemann Problems: Example 1

Consider the following Riemann problem:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

$$u(x,0) = \begin{cases} 1 & x < 0, \\ -1 & x \geq 0. \end{cases}$$



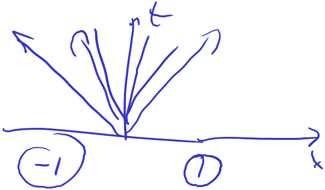
Shock speed: 0

Riemann Problems: Example 2

"rarefaction"

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

$$u(x,0) = \begin{cases} -1 & x < 0, \\ 1 & x \geq 0. \end{cases}$$



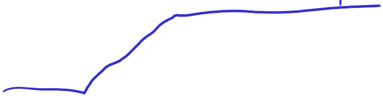
(IC sign flip compared to previous slide)

Both u_1 and u_2 are weak solutions

Shock speed: 0

$u_1(x,t) = u(x,0)$

$$u_2(x,t) = \begin{cases} -1 & x \leq -t \\ x/t & \text{otherwise} \\ 1 & x > t \end{cases}$$



$$-\frac{x}{2t^2} + \left(\frac{x^2}{4t^2}\right)_x = 0$$

Bad Shocks and Good Shocks

In the shock version of the 'ambiguous' Riemann problem, where do the characteristics go?

out of the shock

Comment on the stability of that situation.

"feels unstable" "not self-steepening"

Ad-Hoc Idea: Ban Bad Shocks

Recall: what is $f'(u)$?

characteristic speed

Devise a way to ban unstable shocks.

"Good shock":

$$f'(u^-) > s > f'(u^+) \leftarrow \text{"entropy condition"}$$

shock speed

$f'' \geq 0$

If f is convex, ($\Leftrightarrow f'$ increasing), $f'(u^-) > f'(u^+)$

$\Rightarrow u^- \geq u^+$

$\Leftrightarrow$ entropy condition

Vanishing Viscosity Solutions

Goal: neither uniqueness nor existence poses a problem.

How?

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}$$ with small "$\varepsilon \to 0$"

Define "vanishing viscosity solution":

$$u(x,t) = \lim_{\varepsilon \to 0} u_\varepsilon(x,t).$$

Entropy-Flux Pairs

What are features of (physical) entropy?

- constant along particle paths
- jumps to a higher value across a shock

Definition (Entropy/Entropy Flux)

An entropy $\eta(u)$ and an entropy flux $\psi(u)$ are functions so that η is convex and

$$\eta(u)_t + \psi(u)_x = 0$$

for smooth solutions of the conservation law.

$\eta'' \geq 0$

Finding Entropy-Flux Pairs

(Assume u smooth

$\eta(u)_t + \psi(u)_x = 0$. Find conditions on η and ψ.

chain rule: $\eta'(u)\, u_t + \psi'(u)\, u_x \overset{①}{=} 0$

$u_t + f'(u)\, u_x = 0$

$\eta'(u)\, u_t + \eta'(u) f'(u)\, u_x \overset{②}{=} 0$

$\psi'(u) \overset{①}{=} -\eta'(u) \frac{u_t}{u_x} \overset{②}{=} \eta'(u) f'(u)$

Come up with an entropy-flux pair for Burgers.

$f(u) = \frac{u^2}{2}$, pick $\eta(u) = u^2 \Rightarrow \psi'(u) = 2u \cdot u$

one solution: $\psi(u) = 2u^3/3$

Back to Vanishing Viscosity (1/2)

$$u_t + f(u)_x = \varepsilon u_{xx}$$

What's the evolution equation for the entropy?