Schemes in Conservation Form

Definition (Conservative Scheme)

A conservation law scheme is called conservative iff it can be written as

$$u_{j,\ell+1} = u_{j,\ell} - \frac{\Delta t}{\Delta x}\left(f^*_{j+\frac{1}{2}}(\vec{u}_\ell) - f^*_{j-\frac{1}{2}}(\vec{u}_\ell) \right)$$

where f^*...

- ~~Lipschitz~~ continuous
- "consistency": $f^*(u,u,u,\dots,u) = f(u)$

Theorem (Lax-Wendroff)

If the solution $\{u_{j,\ell}\}$ to a conservative scheme converges (as $\Delta t, \Delta x \to 0$) boundedly almost everywhere to a function $u(x,t)$, then u is a weak solution of the conservation law.

Lax-Wendroff Theorem: Proof

$\Delta_i^{\pm} \to$ applied on index axis i.

Summation by parts: With $\Delta^+ a_k = a_{k+1} - a_k$ and $\Delta^- a_k = a_k - a_{k-1}$:

$\int u' v = [\ ] - \int u v'$

$$\sum_{k=1}^{N} a_k(\Delta^- \varphi_k) + \sum_{k=1}^{N} \varphi_k(\Delta^+ a_k) = -a_1\varphi_0 + \varphi_N a_{N+1}.$$

Let $\varphi_{j,\ell} = \varphi(x_j, t_\ell)$ $\varphi \in C_0^1$ ("compact support").

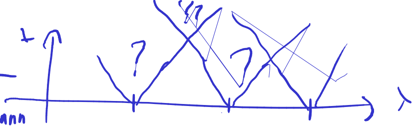
$$0 = \sum_{\ell=1}^{\infty} \sum_j \left(\left(\frac{\Delta_2^+ u_{j,\ell}}{h_t} + \frac{\Delta_1^+ f^*_{j-\frac{1}{2}}}{h_x} \right) \varphi_{j,\ell} \right) h_x h_t \to 0 \text{ by cons.}$$

$$= -\sum_{\ell=1}^{\infty} \sum_j \left(\frac{(\Delta_2^- \varphi_{j,\ell}) u_{j,\ell}}{h_t} + \frac{\Delta_1^- \varphi_{j,\ell}}{h_x} f^*_{j-\frac{1}{2}} \right) h_x h_t - \sum_j u_{j,1} \varphi_{j,0} h_x$$

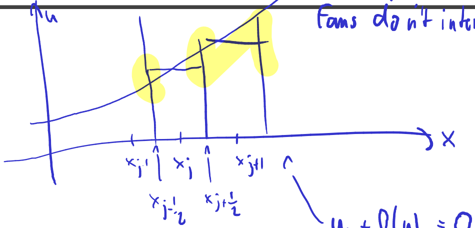
$$\xrightarrow[(h_x, h_t \to 0)]{\text{DCT}} \int_0^\infty \int_{-\infty}^{\infty} \varphi_t u + \varphi_x f(u) \, dx \, dt - \int_{-\infty}^{\infty} u(x,0)\varphi(x,0) \, dx$$

Finite Volume Schemes

Intuitively: Choose Δt so that Riemann Fans don't intersect.

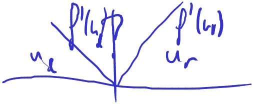


Finite volume: Idea?



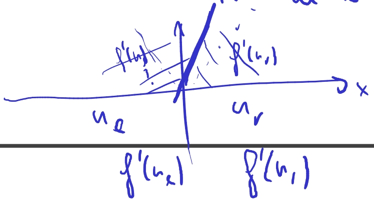
$u_t + f(u)_x = 0$

$$\bar{u}_j = \frac{1}{h_x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x)\,dx$$

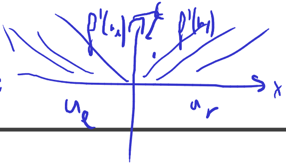


Idea behind FV: all Riemann problems all the time.

shock:



rarefaction:

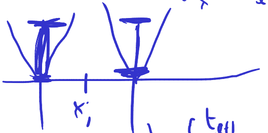


Developing Finite Volume

$$\int_{t_\ell}^{t^{\ell+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} (u_t + f(u)_x)\mathrm{d}x\mathrm{d}t = 0$$

$$\frac{1}{h_y}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^{\ell+1}(x)\,dx - \frac{1}{h_x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_\ell(x)\,dx$$

$$+\frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j+\frac{1}{2}}(t))\,dt - \frac{1}{h_y}\int_{t_\ell}^{t_{\ell+1}} f(u_{j-\frac{1}{2}}(t))\,dt = 0$$



$$\bar{u}_{j,\ell+1} - \bar{u}_{j,\ell}$$

$$+\frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j+\frac{1}{2}}(t))\,dt - \frac{1}{h_y}\int_{t_\ell}^{t_{\ell+1}} f(u_{j-\frac{1}{2}}(t))\,dt = 0$$

Flux Integrals?

$$\frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j+1/2})\,\mathrm{d}t?$$ ← Halp?

Observe: Solutions to Riemann problems are self-similar.

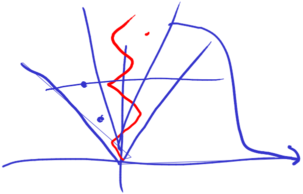
T: $\bar{x} = ax$ $\quad \bar{t} = at$

$$u_t + f(u)_x = 0 \rightsquigarrow \;?\, u_{\bar{t}} + \,?\, f(u)_{\bar{x}} = 0$$

↳ Change of variables T leaves c. law invariant. IC

⇒ Riemann problem

$$u(x,t) = U(x/t)$$



just straight lines

The Godunov Scheme

Altogether:

$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} - \frac{h_t}{h_x}(f(u_{j+1/2,\ell}) - f(u_{j-1/2,\ell})).$$

Overall algorithm?

Heuristic time step restriction?