Harten's Lemma

Theorem (Harten's Lemma)

If a scheme can be written as

$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} + \lambda(C_{j+1/2}\Delta_+\bar{u}_j - D_{j-1/2}\Delta_-\bar{u}_j)$$

with $C_{j+1/2} \geq 0$, $D_{j+1/2} \geq 0$, $1 - \lambda(C_{j+1/2} + D_{j+1/2}) \geq 0$ *and* $\lambda = h_t/h_x$, *then it is TVD.*

As a matter of notation, we have

$$\begin{aligned}\Delta_+ u_j &= u_{j+1} - u_j,\\ \Delta_- u_j &= u_j - u_{j-1}.\end{aligned}$$

$f(a) - f(b) = C \cdot (a-b)$

$f'(y)$

We have omitted the time subscript for the time level ℓ.

Minmod Scheme

Still assume $f'(u) \geq 0$.

$$f^{*,(1)}_{j+1/2} = f\Big(\bar{u}_j + \underbrace{\tfrac{1}{2}(\bar{u}_{j+1} - \bar{u}_j)}_{\tilde{u}_j^{(1)}}\Big), \qquad f^{*,(2)}_{j+1/2} = f\Big(\bar{u}_j + \underbrace{\tfrac{1}{2}(\bar{u}_j - \bar{u}_{j-1})}_{\tilde{u}_j^{(2)}}\Big).$$

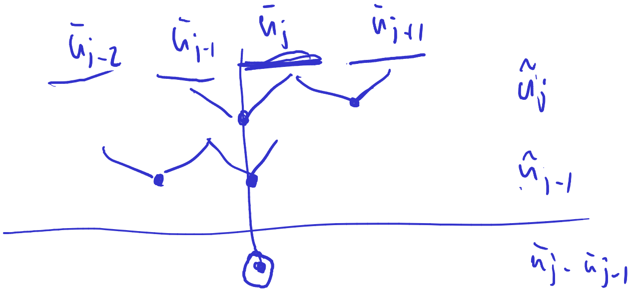
$f^*(u^+, u^-) = f(u^-)$

Design a 'safe' thing to use for $\tilde{u}$:

$$\operatorname{minmod}(a,b) = \begin{cases} a & |a| < |b|,\ ab \geq 0 \\ b & |a| \geq |b|,\ ab \geq 0 \\ 0 & ab < 0 \end{cases} \qquad \hat{u}_j := \operatorname{minmod}\left(\tilde{u}_j^{(1)}, \tilde{u}_j^{(2)}\right)$$

$$f^{*,(3)}_{j+\frac{1}{2}} = f(\bar{u}_j + \hat{u}_j)$$

$$\underbrace{\frac{\hat{u}_j}{\bar{u}_j - \bar{u}_{j-1}}}_{0 \leq \cdot \leq \frac{1}{2}} \qquad \underbrace{\frac{\tilde{u}_{j-1}}{\bar{u}_j - \bar{u}_{j-1}}}_{0 \leq \cdot \leq \frac{1}{2}}$$


$\bar{u}_{j-2}$
$\bar{u}_{j-1}$
$\bar{u}_j$
$\bar{u}_{j+1}$
$\hat{u}_j$
$\hat{u}_{j-1}$
$\bar{u}_j - \bar{u}_{j-1}$

Minmod is TVD

Show that Minmod is TVD:

$$\bar u_{j,\ell+1} = \bar u_{j,\ell} - \lambda \left[f(\bar u_j + \tilde u_j) - f(\bar u_{j-1} + \hat u_{j-1}) \right]$$

$$= \bar u_j - \lambda \left[- D_{j-\frac{1}{2}} \Delta_- \bar u_j \right]$$

$$D_{j-\frac{1}{2}} = \frac{f(\bar u_j + \tilde u_j) - f(\bar u_{j-1} + \hat u_{j-1})}{\bar u_j - \bar u_{j-1}} = f'(\xi) \frac{\bar u_j - \bar u_{j-1} + \tilde u_j - \hat u_{j-1}}{\bar u_j - \bar u_{j-1}}$$

$$= \underbrace{f'(\xi)}_{\substack{\geq 0 \\ \text{(assumption)}}} \left[1 + \underbrace{\frac{\tilde u_j}{\bar u_j - \bar u_{j-1}}}_{0 \leq \cdot \leq \frac{1}{2}} - \underbrace{\frac{\hat u_{j-1}}{\bar u_j - \bar u_{j-1}}}_{0 \leq \cdot \leq \frac{1}{2}} \right] \overset{!}{\geq} 0$$

Minmod: CFL restriction?

Derive a time step restriction for Minmod.

$$D_{j-\frac{1}{2}} \leq \frac{3}{2} f'(\xi) \leq \frac{3}{2} \max |f'(u)|$$

$$0 \overset{!}{\leq} 1 - \lambda D_{j-\frac{1}{2}} \geq 1 - \frac{3}{2}\lambda \max_u |f'(u)| \overset{!}{\geq} 0 \Leftrightarrow \lambda \leq \frac{2}{3} \cdot \frac{1}{\max |f'(u)|}$$

→ monotonic?

↑ upwind

MUSCL

↳ real

↳ cons..

What about Time Integration?

SSP (strongly stability preserving) SSPRK(2,2)

$$u^{(1)} = u_\ell + h_t L(u_\ell), \qquad u_{\ell+1} = \frac{u_\ell}{2} + \frac{1}{2}(u^{(1)} + h_t L(u^{(1)})).$$

Above: A version of RK2 with L the ODE RHS. Will this cause wrinkles? $\alpha \in [0,1]$

TV is a convex functional: $TV(\alpha \vec{u} + (1-\alpha)\vec{v}) \le \alpha TV(\vec{u}) + (1-\alpha) TV(\vec{v})$

$$\begin{aligned}
TV(u_{\ell+1}) &= TV\left(\frac{u_\ell}{2} + \frac{1}{2}\left(u^{(1)} + h_t L(u^{(1)})\right)\right)\\
&\le \frac{1}{2} TV(u_\ell) + \frac{1}{2} TV\left(u^{(1)} + h_t L(u^{(1)})\right)\\
&\le \frac{1}{2} TV(u_\ell) + \frac{1}{2} TV(u^{(1)})\\
&\le \frac{1}{2} TV(u_\ell) + \frac{1}{2} TV(u_\ell)\\
&\le TV(u^\ell)
\end{aligned}$$

$u^{\ell+1} = u_\ell - \lambda(\quad u_\ell)$

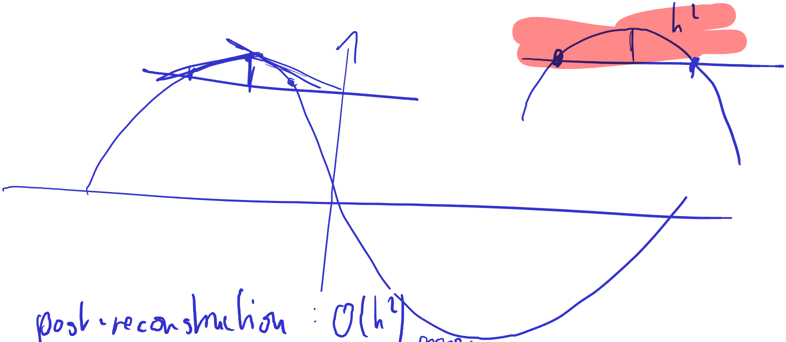
Total Variation is Convex

Show: $TV(\cdot)$ is a convex functional.

$$TV(\alpha\vec{u} + (1-\alpha)\vec{v})$$
$$= \sum_j \left| \alpha(u_j - u_{j-1}) + (1-\alpha)(v_j - v_{j-1}) \right|$$
$$\leq \alpha \sum_j |u_j - u_{j-1}| + (1-\alpha) \sum_j |v_j - v_{j-1}|$$
$$\leq \alpha\, TV(\vec{u}) + (1-\alpha)\, TV(\vec{v})$$

TVD and High Order

Can TVD schemes be high order everywhere? (aside from near shocks)



post-reconstruction : $O(h^2)$ error

post-flux : $O(h)$ [Osher, C. '84]

High Order at Smooth Extrema

$p_1(x) = 1$
$p_2(x) = (x - x_1)$
$p_3(x) = (x - x_1)(x - x_2)$

- TVB Schemes [Shu '87]
- ENO [Harten/Engquist/Osher/Chakravarthy '87]
  - Define $W_j = w(x_{j+1/2}) = \int_{x_{1/2}}^{x_{j+1/2}} u(\xi, t) d\xi = h_x \sum_{i=1}^{j} \bar{u}_i$
    - Observe $u_{j+1/2} = w'(x_{j+1/2})$.
    - Approximate by interpolation/numerical differentiation.
  - Start with the linear function $p^{(1)}$ through W_{j-1} and W_j
  - Compute divided differences on (W_{j-2}, W_{j-1}, W_j)
  - Compute divided differences on (W_{j-1}, W_j, W_{j+1})
  - Use the one with the smaller magnitude (of the divided differences) to extend $p^{(1)}$ to quadratic
  - (and so on, adding points on the side with the lowest magnitude of the divided differences)
- WENO [Liu/Osher/Chan '94]

Outline