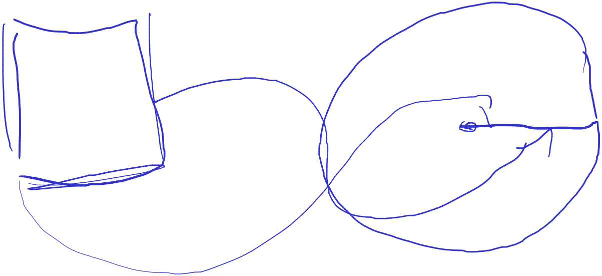
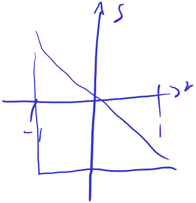
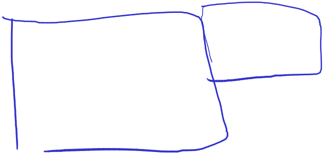
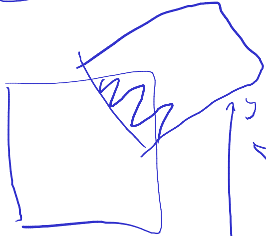
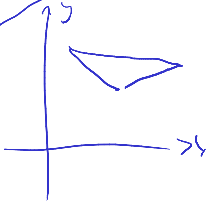


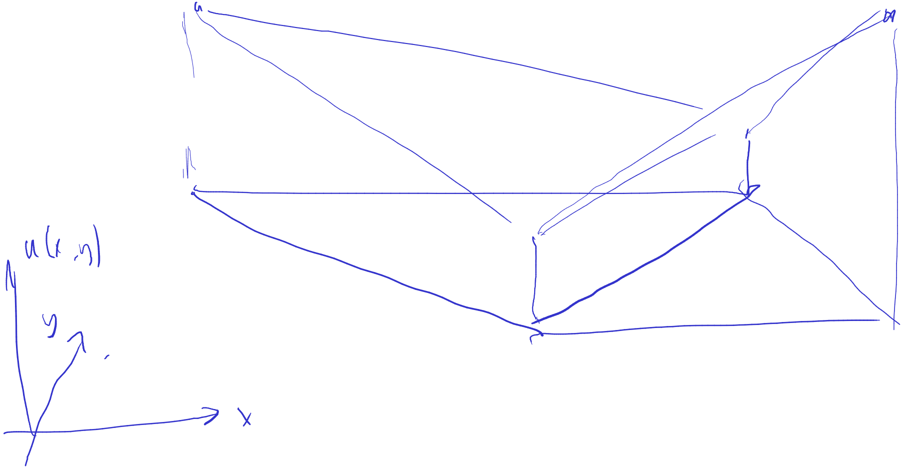




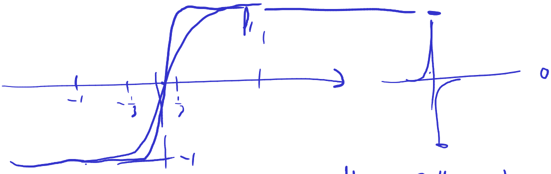




$$\psi\begin{pmatrix} r\\ s\end{pmatrix} = A\begin{pmatrix} r\\ s\end{pmatrix} + \vec{b} = \begin{pmatrix} x\\ y\end{pmatrix}$$


u(x, y)
y
x

Function Spaces



Consider

$$f_n(x) = \begin{cases} -1 & x \le -\frac{1}{n}, \\ \frac{3n}{2}x - \frac{n^3}{2}x^3 & -\frac{1}{n} < x < \frac{1}{n}, \\ 1 & x \ge 1/n. \end{cases}$$

$\|f_n - f\|_\infty \not\to 0$

$= 1$

$\|f_n - f\|_1 \to 0$

Converges to the step function. Problem?

Lose all our smoothness

Norms

Definition (Norm)

A norm $\|\cdot\|$ maps an element of a *vector space* into $[0, \infty)$. It satisfies:

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Convergence

Definition (Convergent Sequence)

$x_n \to x :\Leftrightarrow \|x_n - x\| \to 0$ (convergence in norm)

Definition (Cauchy Sequence)

For all $\varepsilon > 0$, there exists an n for which

$\|x_\nu - x_\mu\| < \epsilon$ for $\mu, \nu \geq n$.

Banach Spaces

Definition (Complete/"Banach" space)

Cauchy ⇒ convergent.

What's special about Cauchy sequences?

Limits for free!

Counterexamples?

- C^0 or C^1 with $\|\cdot\|_1$ / $\|\cdot\|_2$
- $\mathbb{Q}$ with $|\cdot|$

More on C^0

Let $\Omega \subseteq \mathbb{R}^n$ be open. Is $C^0(\Omega)$ with $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$ Banach?

$f(x) = \frac{1}{x}$ $\Omega = (0,1)$ "$\|f\| = \infty$" not allowed.

Is $C^0(\bar{\Omega})$ with $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$ Banach?

Assume $\{f_i\}$ Cauchy.

- For each $x \in \bar{\Omega}$, $(f(x))_i \subset \mathbb{R}$ is Cauchy $\Rightarrow$ pointwise limit $\exists$. Call the result f.
- Let $\varepsilon > 0$. There exists an N so that $|f_n^{(x)} - f_m^{(x)}| < \varepsilon$ for all $m, n \geq N$ for all $x \in \bar{\Omega}$. Take limit $m \to \infty$. $|\underline{f_n(x) - f(x)}| < \varepsilon$ holds pw. $\underline{\text{and}}$ for all $x \in \bar{\Omega}$. $\|f_n - f\|_\infty < \varepsilon$. Conv. in $\|\cdot\|_\infty$ is called "uniform conv.", preserving continuity.

C^m Spaces

Let $\Omega \subseteq \mathbb{R}^n$.

Consider a multi-index $\boldsymbol{k} = (k_1, \ldots, k_n)$ $\in \mathbb{N}_0^n$ and define the symbols

$$D^{\vec{k}} f = \frac{\partial^{|k|_1}}{\partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n}} f. \qquad |k| = k_1 + \ldots + k_n$$

Definition (C^m Spaces)

$$C^m(\Omega) = \{ f \in C^0(\Omega) : D^k f \in C^0 \text{ for all } \vec{k} \text{ w/ } |k| \le m \}$$

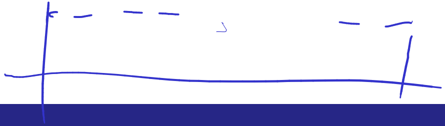
$$C^\infty(\Omega) = \{ f \in C^0(\Omega) : D^k f \in C^0 \text{ for all } \vec{k} \}$$

$$C_0^m(\Omega) = \{ f \in C^m(\Omega) : f \text{ has compact support:}$$

$\exists\, \Omega' \subset \Omega$: closed, bdd ("compact") so that $f|_{\Omega \setminus \Omega'} \equiv 0 \}$

L^p Spaces

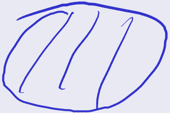
Let $1 \leq p < \infty$.



Definition (L^p Spaces)

$$L^p(\Omega) := \left\{ u : (u : \mathbb{R} \to \mathbb{R}) \text{ measurable}, \int_\Omega |u|^p \, dx < \infty \right\},$$

$$\|u\|_p := \left(\int_\Omega |u|^p \, dx \right)^{1/p}.$$



Definition (L^∞ Space)

$$L^\infty(\Omega) := \{ u : (u : \mathbb{R} \to \mathbb{R}), |u(x)| < \infty \text{ almost everywhere} \},$$

$$\|u\|_\infty = \inf \{ C : |u(x)| \leq C \text{ almost everywhere} \}.$$

↳ everywhere but a set of zero "volume."

L^p Spaces: Properties

$\|s\hat{u}\|_{L^1(-\pi,\pi)} \leq \|s\|_\infty \|\hat{u}\|_1$

Theorem (Hölder's Inequality)

For $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$ and measurable u and v,

$$\boxed{\|uv\|_1 \leq \|u\|_p \|v\|_q}$$

Theorem (Minkowski's Inequality (Triangle inequality in L^p))

For $1 \leq p \leq \infty$ and $u, v \in L^p(\Omega)$,

$$\boxed{\|u+v\|_p \leq \|u\|_p + \|v\|_p}$$

Inner Product Spaces

Let V be a vector space.

Definition (Inner Product)

An inner product is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that for any $f, g, h \in V$ and $\alpha \in \mathbb{R}$

$$\begin{aligned} \langle f, f \rangle &\geq 0, \\ \langle f, f \rangle &= 0 \Leftrightarrow f = 0, \\ \langle f, g \rangle &= \langle f, g \rangle, \\ \langle \alpha f + g, h \rangle &= \alpha \langle f, h \rangle + \langle g, h \rangle . \end{aligned}$$

Definition (Induced Norm)

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

Hilbert Spaces

Definition (Hilbert Space)

An inner product space that is complete under the induced norm.

Let Ω be open.

Theorem (L^2)

$L^2(\Omega)$ equals the closure of (set of all limits of Cauchy sequences in) $C_0^\infty(\Omega)$ under the induced norm $\|\cdot\|_2$.

Theorem (Hilbert Projection (e.g. Yosida '95, Thm. III.1))

$M \subseteq V$ closed subspace of a Hilbert space V. Let $u \in V$,
There exists a unique $v \in M$ $u = v + w$ $w \in M^\perp$

$M^\perp = \{ w \in V : (z, w)_V = 0 \text{ for all } z \in M \}$

Weak Derivatives

Define the space L^1_{loc} of locally integrable functions.

Definition (Weak Derivative)

$v \in L^1_{\text{loc}}(\Omega)$ is the weak partial derivative of $u \in L^1_{\text{loc}}(\Omega)$ of multi-index order $\boldsymbol{k}$ if

In this case, $D^{\boldsymbol{k}} u := v$