A Boundary Value Problem

Consider the following elliptic PDE

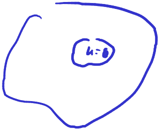
$$-\nabla \cdot (\kappa(\boldsymbol{x})\nabla u) = f(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega \subset \mathbb{R}^2,$$
$$u(\boldsymbol{x}) = 0 \quad \text{when} \quad \boldsymbol{x} \in \partial\Omega.$$

Weak form?

$$\underbrace{\int_\Omega \kappa \, \nabla u \cdot \nabla v}_{a(u,v)} = \underbrace{\int_\Omega f v}_{g(v)}$$

Weak Form: Bilinear Form and RHS Functional

Hence the problem is to find $u \in V$, such that



$$a(u,v) = g(v), \quad \text{for all } v \in V = H_0^1(\Omega)$$

where. . .

$$\underbrace{\int_\Omega \kappa(x) \nabla u \cdot \nabla v}_{a(u,v)} = \underbrace{\int_\Omega f v}_{g(v)}$$

Is this symmetric, coercive, and continuous?

symm: ✓

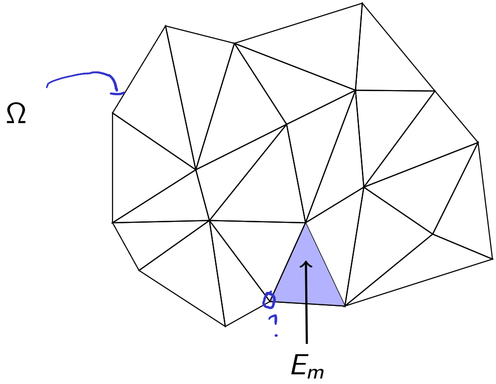
coercivity: $c_1 \|u\|^2_{H^1} \le a(u,u) \Leftarrow 0 < c \le \kappa(x) \ \forall x$

Lax-M says yeah!

continuity: $a(u,v) \le \|u\|_{H^1} \|v\|_{H^1} \Leftarrow \kappa(x) \le C < \infty \ \forall x$

Triangulation: 2D

Suppose the domain is a union of triangles E_m, with vertices x_i.



$$\bar{\Omega} = \bigcup_{i=1}^{M} E_m.$$

Elements and the Bilinear Form

If the domain, Ω, can be written as a disjoint union of elements, E_k,

$$\Omega = \cup_{m=1}^{M} E_m \quad \text{with} \quad E_i^\circ \cap E_j^\circ = \emptyset \text{ for } i \neq j,$$

what happens to a and g?

$$a(u,v) = \sum_{m=1}^{M} \int_{E_m} K(x) \cdot \nabla u \, \nabla v$$

$$g(v) = \sum_{m=1}^{M} \int_{E_m} F v$$

Basis Functions

Expand

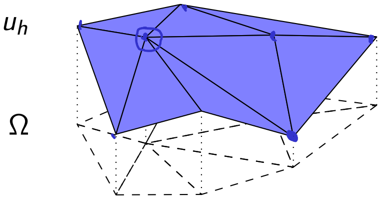
$$u_N(\boldsymbol{x}) = \sum_{i=1}^{N_p} u_i \varphi_i, (\vec{x})$$

and plug into the weak form.

$$a(u_N, \varphi_i) = \sum_{j=1}^{N_p} u_j \, a(\varphi_j, \varphi_i) \qquad i = 1 \ldots N_p$$

Global Lagrange Basis

Approximate solution u_h: Piecewise linear on Ω

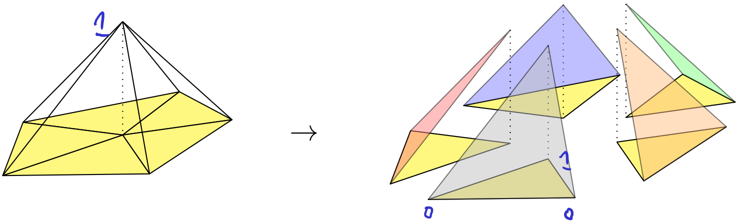


The Lagrange basis for V_h consists of piecewise linear φ_i, with...

Basis Functions Features

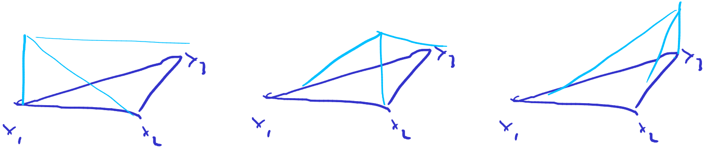
Features of the basis?

- u_N must be continuous (C^0) because basis is.
- Restricted to E_m, each φ_i is linear.



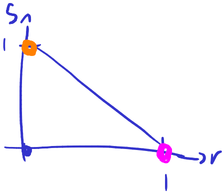
Local Basis

What basis functions exist on each triangle?



Local Basis Expressions

Write expressions for the nodal linear basis in 2D.

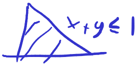


$\varphi_1(r,s) = 1 - r - s$

$\varphi_2(r,s) = r$

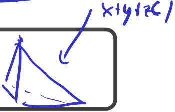
$\varphi_3(r,s) = s$

Higher-Order, Higher-Dimensional Simplex Bases



What's an *n*-simplex?

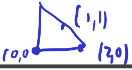
$$r_i \geq 0 \qquad \sum_i r_i \le 1$$



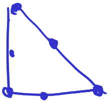
Give a higher-order polynomial space on the *n*-simplex:

$$P^N = \operatorname{span}\left\{ \prod_{i=1}^{d} r_i^{n_i} : \sum n_i \le N \right\}$$

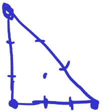
P^2 in 2D: $\quad 1, r, s, r^2, s^2, rs$



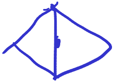
Give nodal sets (on the $\triangle$) for P^N and dim P^N in general.



$P2$



P^3

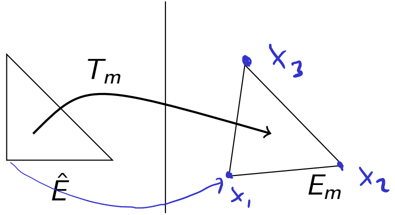


Finding a Nodal/Lagrange Basis in General

Given a nodal set $(\xi_i)_{i=1}^{N_p} \subset \hat{E}$ (where $\hat{E}$ is the reference element) and a basis $(\varphi_j)_{j=1}^{N_p} : \hat{E} \to \mathbb{R}$, find a Lagrange basis.

(do a linear solve)

Element Mappings



$A\vec{x} + \vec{b}$

Construct a mapping $T_m : \hat{E} \to E_m$. Reference element $\hat{E}$, global $\triangle$ E_m.

$$T_m\underbrace{(r, s)}_{=\vec{r}^T} = (\vec{x}_2 - \vec{x}_1)\, r + (\vec{x}_3 - \vec{x}_1)\, s + \vec{x}_1$$

What is the Jacobian of T_m?

$$J_T = \begin{pmatrix} \vec{x}_2 - \vec{x}_1 & \vec{x}_3 - \vec{x}_1 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

Constructing the Global Basis

$\int \nabla u \nabla v \sim \int \nabla \varphi_i \nabla \varphi_j$

Construct a basis on the element E_m from the reference basis $(\cancel{\varphi}_j)_{j=1}^{N_p} : E_m \to \mathbb{R}$.

$$\varphi_j(\vec{x}) = \hat{\varphi}_j(\underbrace{T_n^{-1}(\vec{x})}_{\vec{r}})$$

What's the gradient of this basis?

$$\nabla_{\vec{x}} \varphi_j(T^{-1}(\vec{x})) = \left[\frac{d}{dx} \varphi_j(T^{-1}(\vec{x})) \right]^T$$

$$= \left[\left.\frac{d\varphi_j}{d\vec{r}}\right|_{T^{-1}(\vec{x})} J_T^{-1}(\vec{x}) \right]^T$$

$$= J_T^{-T}(\vec{x}) \, \nabla_{\vec{r}} \varphi_j(T^{-1}(\vec{x}))$$

Assembling a Linear System

Express the matrix and vector elements in

$$\sum_{j=1}^{N_p} u_j a(\varphi_j, \varphi_i) = g(\varphi_i) \quad \text{for } i = 1, \ldots, N_p.$$

$$a(\varphi_j, \varphi_i) = \sum_{m=1}^{M} \int_{E_m} k(x) \, \nabla \varphi_i(x) \cdot \nabla \varphi_j(x) \, dx$$

$$g(\varphi_i) = \sum_{m=1}^{M} \int_{E_m} F(x) \, \varphi_i(x) \, dx$$

Integrals on the Reference Element

Evaluate

$$\int_E \kappa(\mathbf{x}) \nabla_{\mathbf{x}} \varphi_i(\mathbf{x})^T \nabla_{\mathbf{x}} \varphi_j(\mathbf{x}) d\mathbf{x}.$$

$$= \int_E \kappa(x) \underbrace{(J_T^{-T} \nabla_r \varphi_i)^T (J_T^{-T} \nabla_r \varphi_j)} dx$$

$$\rightarrow \quad \overset{P^1}{=} \underbrace{(J_T^{-T} \nabla_r \varphi_i)^T (J_T^{-T} \nabla_r \varphi_j)} \int_E \kappa(x)\, dx$$

And now the RHS functional.

$$\rightarrow \quad \int_E f(x) \varphi_i(x)\, dx = |J_T| \int_{\hat{E}} f(T(r)) \hat{\varphi}_i(r)\, d\vec{r}$$

Inhomogeneous Dirichlet BCs

Handle an inhomogeneous boundary condition $u(\boldsymbol{x}) = \eta(\boldsymbol{x})$ on $\partial\Omega$.

$u \in H^1_0$?

- Find $u^0 \in H^1(\Omega)$ with $u^0(x) = \eta(x)$ on $\partial\Omega$
- Define $\hat{u} = u - u^0 \in H^1_0$! (phew!)

$u = \hat{u} + u^0$

$a(\hat{u} + u^0, v) = a(\hat{u}, v) + \underline{\underline{a(u^0, v)}} = g(v)$

$\longrightarrow a(\hat{u}, v) = g(v) - a(u^0, v)$

Demo

- **Demo:** Developing FEM in 2D [cleared]
- **Demo:** 2D FEM Using Firedrake [cleared]
- **Demo:** Rates of Convergence [cleared]

Outline

Introduction

Finite Difference Methods for Time-Dependent Problems

Finite Volume Methods for Hyperbolic Conservation Laws

Finite Element Methods for Elliptic Problems
- tl;dr: Functional Analysis
- Back to Elliptic PDEs
- Galerkin Approximation
- Finite Elements: A 1D Cartoon
- Finite Elements in 2D
- Approximation Theory in Sobolev Spaces
- Saddle Point Problems, Stokes, and Mixed FEM
- Non-symmetric Bilinear Forms

Discontinuous Galerkin Methods for Hyperbolic Problems