W January 18:

- Intro to the Course
- Scope, what to expect
- PDEs and some basics

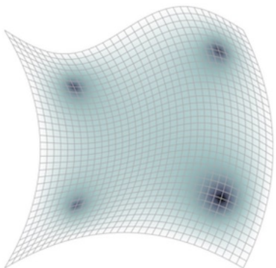
Iterative and Multigrid Methods

CS555 :: Spring 2023

- **Class Time**: Monday/Wednesday 11:00am-12:15pm Catalog
- **Class Location**: 1035 Campus Instructional Facility (CIF)
- **Class URL**: go.illinois.edu/cs555
- **Slack**: cs555-s23
- **Instructor**: Luke Olson
- **Office Hours**: TBD

About the Course

Are you interested in the numerical approximation of solutions to partial differential equations? **Then this course is for you!**



Goal: Find $u \in V$

such that

$$\mathcal{L}u = f \quad \text{in } \Omega \times [0,T]$$

+ initial conditions

+ boundary conditions

Example

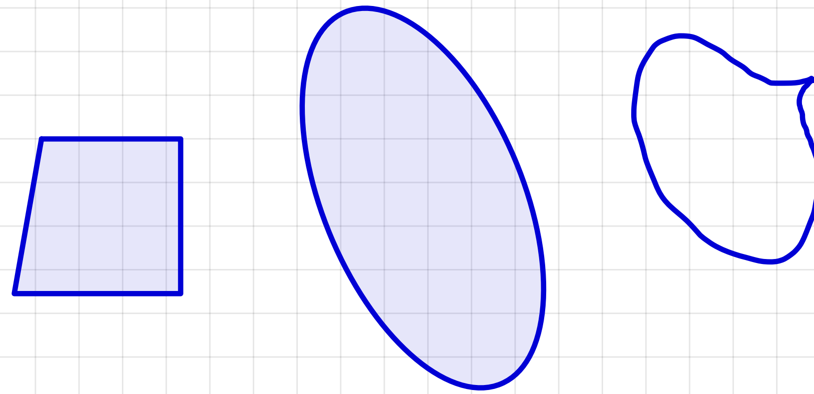
① $-\partial_{xx} u = f \quad \rightarrow \quad \mathcal{L} = -\partial_{xx}$

② $\dfrac{\partial u}{\partial t} + a \dfrac{\partial u}{\partial x} = 0 \quad \rightarrow \quad \mathcal{L} = \dfrac{\partial}{\partial t} + a \dfrac{\partial}{\partial x}$

$f = 0$

Many Questions

Q What is Ω?



Q What is V?

↳ continuous?

↳ smooth?

↳ disc. pw cubic?

Q What does $\frac{\partial}{\partial x}$ mean?

here

here

Q Does a solution exist for

$$\mathcal{L}u = f \quad + \quad IC + BC \; ?$$

Generally, yes, we assume so.

careful!

Q Is it unique? 🤷‍♂️

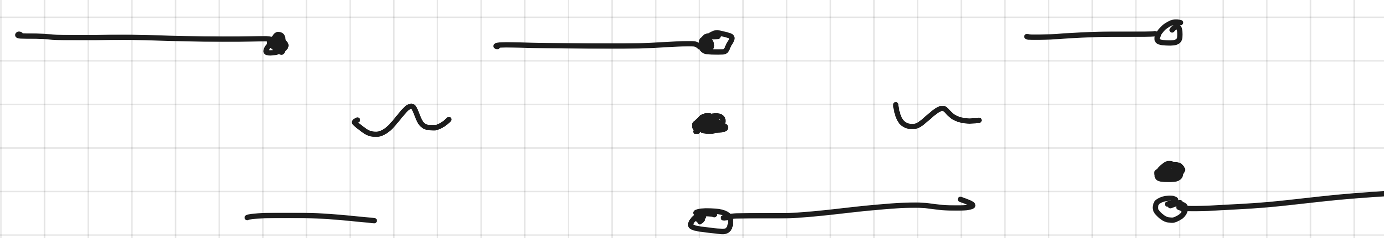
(*) Pure Neumann: $-\partial_{xx}u = f$

$\partial_x u = 0$ at $x=0,1$

$$-\partial_{xx}u - \partial_{yy}u = f$$

$n\cdot\nabla u$ on $\partial\Omega$

(**)



Q Can we find the solution?

① For "model" problems, yes

② For real problems, no

③ Often, we work backwards:

a. define your own u

e.g. $u = 4\sin(3\pi x) + 2$

b. drop into the PDE, let

$$f = \mathcal{L}u$$

② need approximation methods

This course: approximating solutions

0. some basics
1. FD schemes
2. FV schemes
3. FE schemes

HW Roughly bi-weekly → LaTeX
→ Python

Project will develop over the semester

Resources:

Numerical Partial Differential Equations

James Adler
Tufts University
jadler.math.tufts.edu

Hans De Sterck
University of Waterloo
uwaterloo.ca/scholar/hdesterc

Scott MacLachlan
Memorial University of Newfoundland
www.math.mun.ca/~smaclachlan

Luke Olson
University of Illinois Urbana-Champaign
lukeo.cs.illinois.edu

January 18, 2023

- Post this week
- Do not distribute 📖

Definition 1.2: Order of a PDE

The order *of a PDE is the order of the highest-order partial derivative present in the PDE.*

For example, PDE

$$u_{xx} + uu_{yy} + u_x = f(x, y), \tag{1.6}$$

is second-order, while

$$u_t + u_x = f(x, t), \tag{1.7}$$

is first-order.

Definition 1.3: Quasi-Linear PDE

A nonlinear PDE in u in which the highest order partial derivatives appear linearly with coefficients only depending on the lower-order derivatives of u and the independent variables $\boldsymbol{x}$, is called a quasi-linear *PDE.*

For example, the PDE

$$xu^2\, u_{xx} + (u_x + y)\, u_{yy} + y\, u_y^3 = f(x, y), \tag{1.8}$$

is quasi-linear since u_{xx} and u_{yy} both appear linearly. Clearly, the PDE is nonlinear because of the u^2 factor and the u_x and u_y^3 terms.

Definition 1.4: Semi-Linear PDE

A nonlinear PDE is called semi-linear *if it is quasi-linear and if the highest-order terms have coefficients that depend only on the independent variables* $\boldsymbol{x}$.

For example, the following PDE is semi-linear:

$$(x + y + z)\, u_x + z^2\, u_y + \sin(x)\, u_z + u^2 = f(x, y, z). \tag{1.9}$$

First order PDEs

Scalar u $\quad (u(x)$ or $u(x,y,z))$

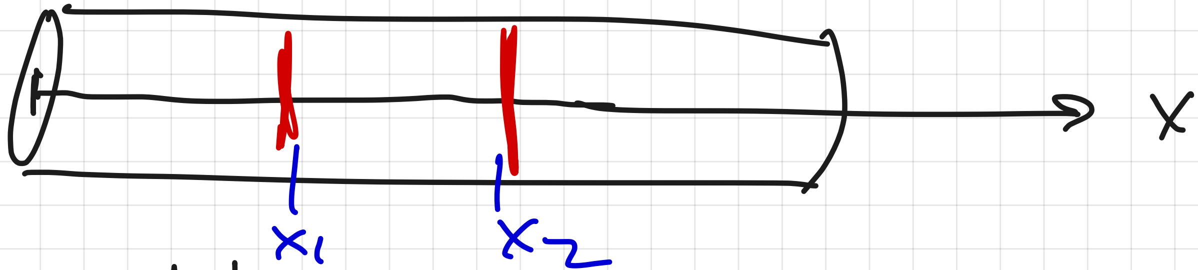
First order conservation law:

$$\frac{\partial u}{\partial t} + \nabla \cdot \underline{f}(u) = 0$$

flux!

$$\underline{f}(u) = \begin{bmatrix} f(u) \\ g(u) \\ h(u) \end{bmatrix}$$

(1D)



x = distance

$\rho(x,t)$ = density of gas.

Mass of the gas in $[x_1, x_2]$ at time t

$$= \int_{x_1}^{x_2} \rho(x,t)\, dx$$

↳ change in mass is only at the ends:

$v(x,t)$ = velocity of the gas

rate of flow (or flux of gas):

$$\rho(x,t)\, v(x,t)$$

change in mass

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = \rho(x_1,t)v(x_1,t) - \rho(x_2,t)\,v(x_2,t)$$

integrate: $t_1 \to t_2$

$$\underbrace{\int_{x_1}^{x_2} \rho(x,t_2)\,dx - \int_{x_1}^{x_2} \rho(x,t_1)\,dx}_{\int_{x_1}^{x_2}\int_{t_1}^{t_2} \frac{\partial}{\partial t}\rho(x,t)\,dt\,dx} = \int_{t_1}^{t_2} \rho(x_1,t)v(x_1,t)\,dt - \int_{t_1}^{t_2} \rho(x_2,t)\,v(x_2,t)\,dt$$

$$= \int_{t_1}^{t_2}\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\rho(x,t)v(x,t)\right) dx\,dt$$

$$\to \rho_t + (\rho v)_x = 0 \quad \rightsquigarrow \rho v = f(v)$$

$$\Downarrow$$

$$\rho_t + (f(v))_x = 0$$

$$\rho_t + (f(\rho))_x = 0$$

Simple form

$u =$ concentration

$$u_t + a u_x = 0 \quad (\text{linear advection})$$

constant $a > 0$

Also

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)u$$

Simplest form: let u = concentration (density)

$$u_t + a\,u_x = 0$$ (linear advection)

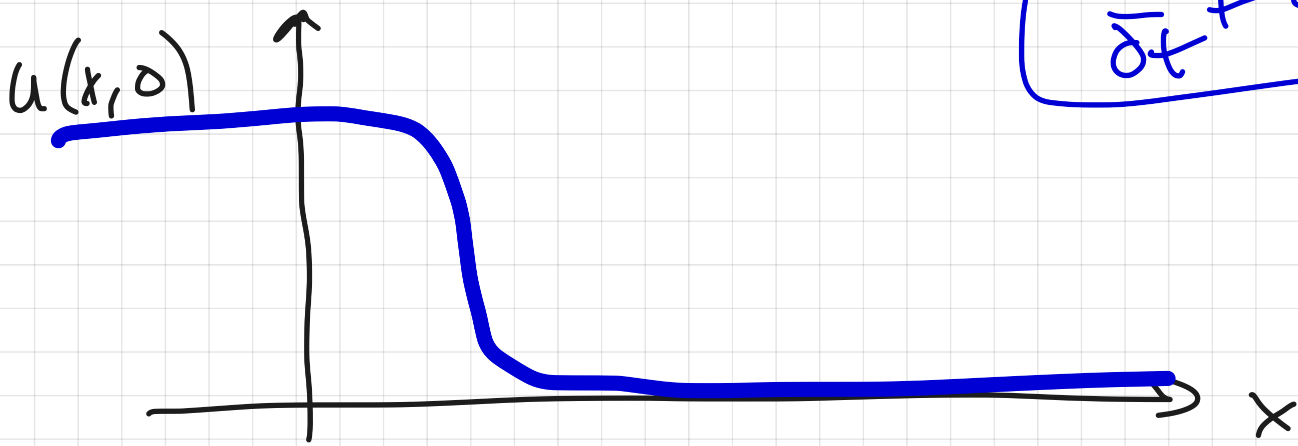
$\uparrow$ constant $a>0$

Also: wave equation:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)u$$

Take initial profile:

$$\boxed{\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0}$$



Think about a path $x(t)$ when u is constant: $\dfrac{d\, u(x(t),t)}{dt} = 0$

$$\downarrow$$

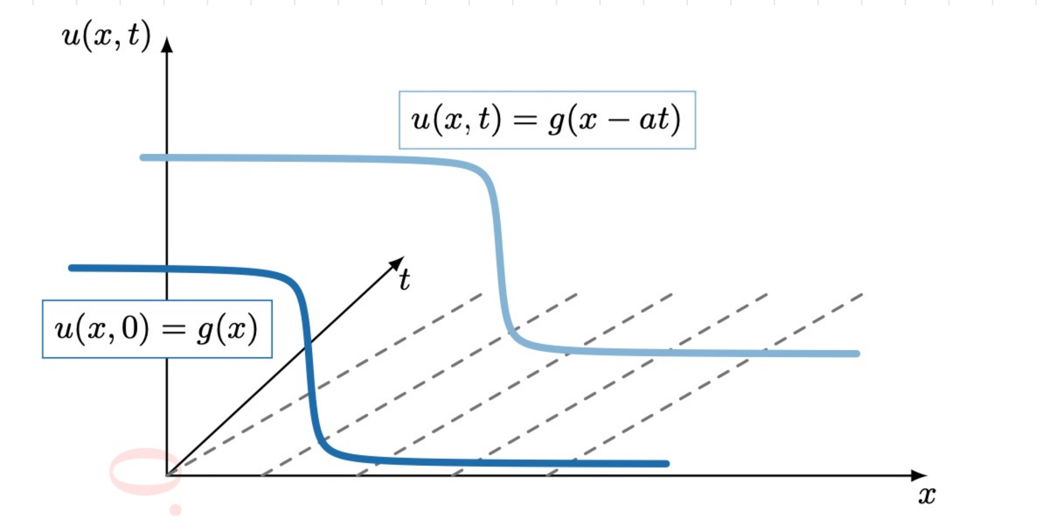
$$\frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial t} = 0$$

$\rightarrow$ happens when $\dfrac{dx}{dt} = a$

$$\rightarrow \underbrace{x(t) = x_0 + at}_{\text{define characteristic curves.}}$$

if $u(x,0) = g(x)$

then $u(x,t) = g(x - at)$



Later, we'll look at hyperbolic conservation laws.

2^{nd} order PDEs :

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y), \quad (1.19)$$

	Criteria	Classification
(i)	$b^2 - ac < 0$	elliptic
(ii)	$b^2 - ac = 0$	parabolic
(iii)	$b^2 - ac > 0$	hyperbolic

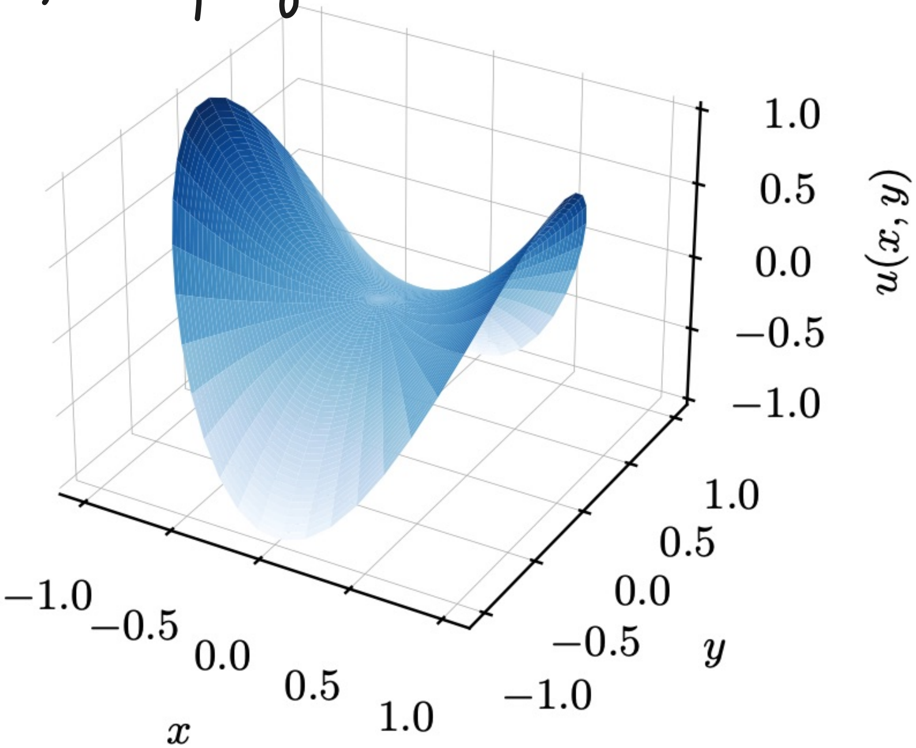
Operator	Common name	Classification
$u_{xx} + u_{yy}$	2D Laplace operator	elliptic
$u_t - u_{xx}$	1D heat (or diffusion) operator	parabolic
$u_{tt} - u_{xx}$	1D wave operator	hyperbolic

Elliptic

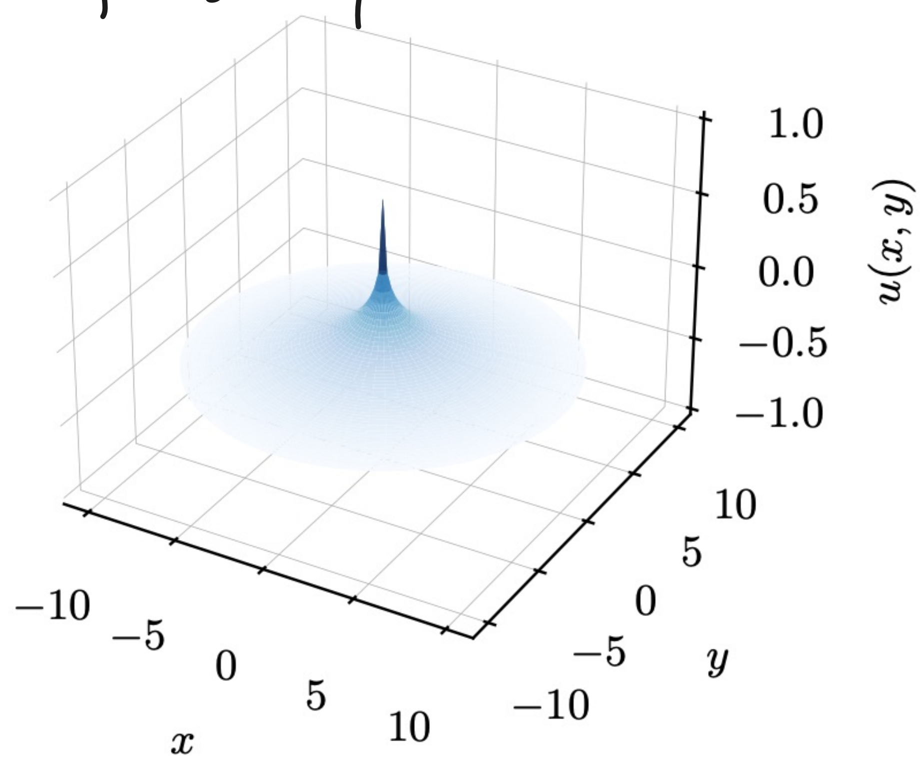
- no characteristic direction, global

$$\text{(BVP)} \qquad \begin{cases} -\nabla \cdot \nabla u &= f(x) \quad \text{in } \Omega, \\ u &= g(x) \quad \text{on } \partial\Omega. \end{cases}$$

$f = 0$, $g = \cos(2\theta)$



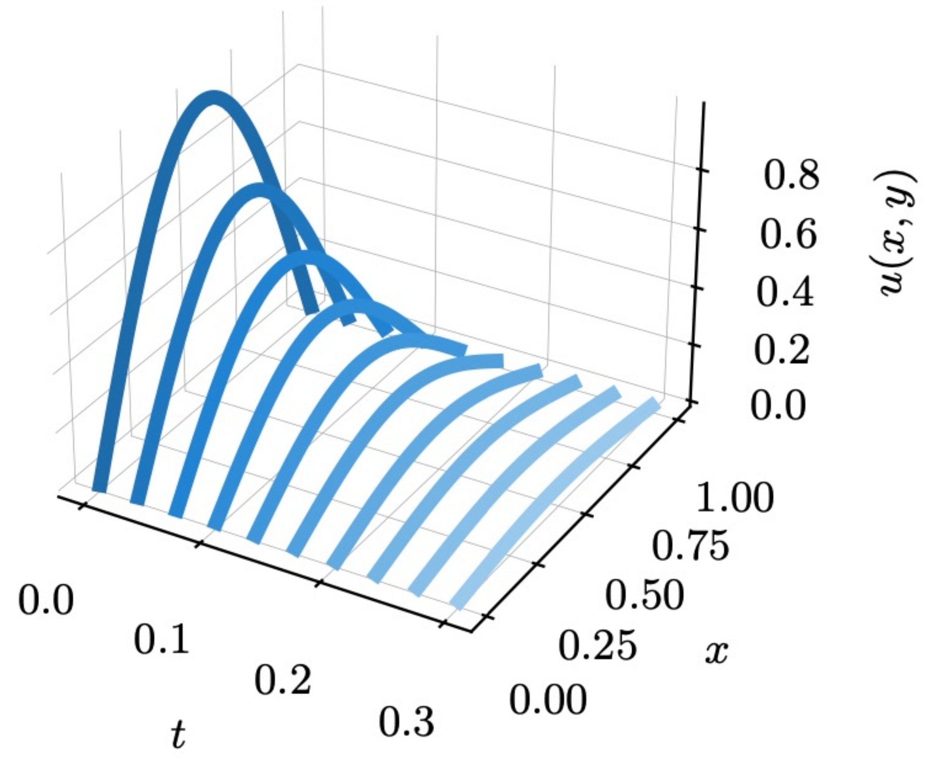
$f = \delta(0)$, $u = 0$ as $r \to \infty$



Parabolic

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 && \text{in } [0,1] \times [0,T] \\ u(x,0) &= \sin(\pi x) && \text{in } [0,1], \\ u(x,t) &= 0 && \text{at } x = 0, 1. \end{aligned}$$

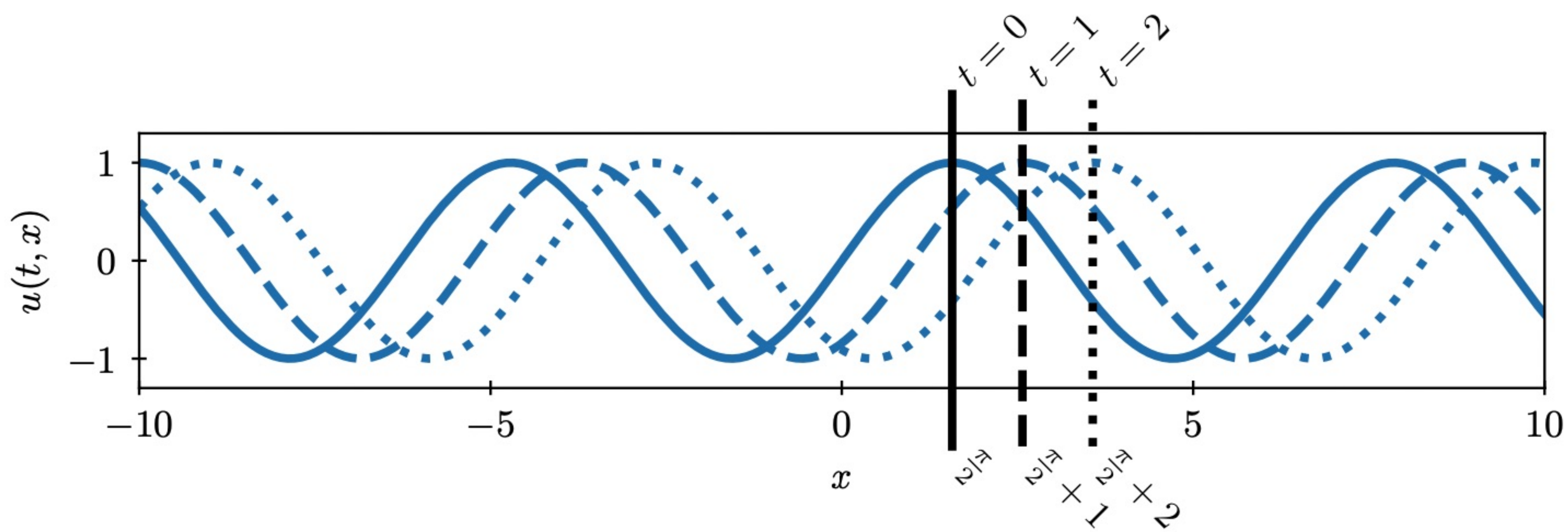
$\hookrightarrow u(x,t) = e^{-\pi^2 t} \sin(\pi x)$



Hyperbolic

$$u_{tt} = c^2 u_{xx}$$

$u(x,0) = g(x) = \sin(x)$



- 1752 — wave equation, d’Alembert
- 1755 — incompressible flow and Euler equation, Euler
- 1759 — wave equation, Euler
- 1762 — wave equation, Bernoulli
- 1775 — differential geometry and the Monge-Ampére equation, Monge
- 1780 — Laplace equation, Laplace
- 1810 — heat equation, Fourier
- 1813 — electromagnetics and the Poisson equation, Poisson
- 1821 — elasticity, Navier, Cauchy
- 1822 — fluid flow and the Navier-Stokes equation, Navier
- 1831 — fluid flow and the Navier-Stokes equation, Poisson
- 1845 — fluid flow and the Navier-Stokes equation, Stokes
- 1860 — acoustics and the Helmholz equation
- 1865 — electromagnetics and Maxwell’s equation, Maxwell
- 1896 — water waves and the Korteweg-De Vries equation