Monday 01/25:

Topic: Finite Differences for time dep problems

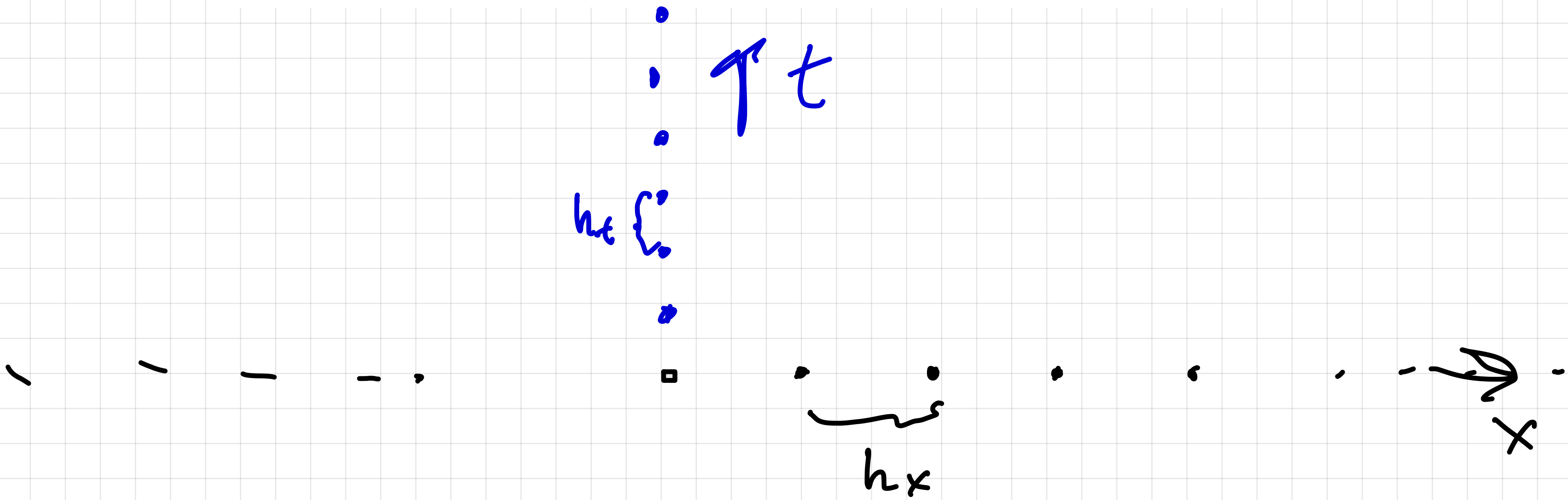
Objectives

1. Introduce explicit and implicit methods

2. Develop 2-level scheme

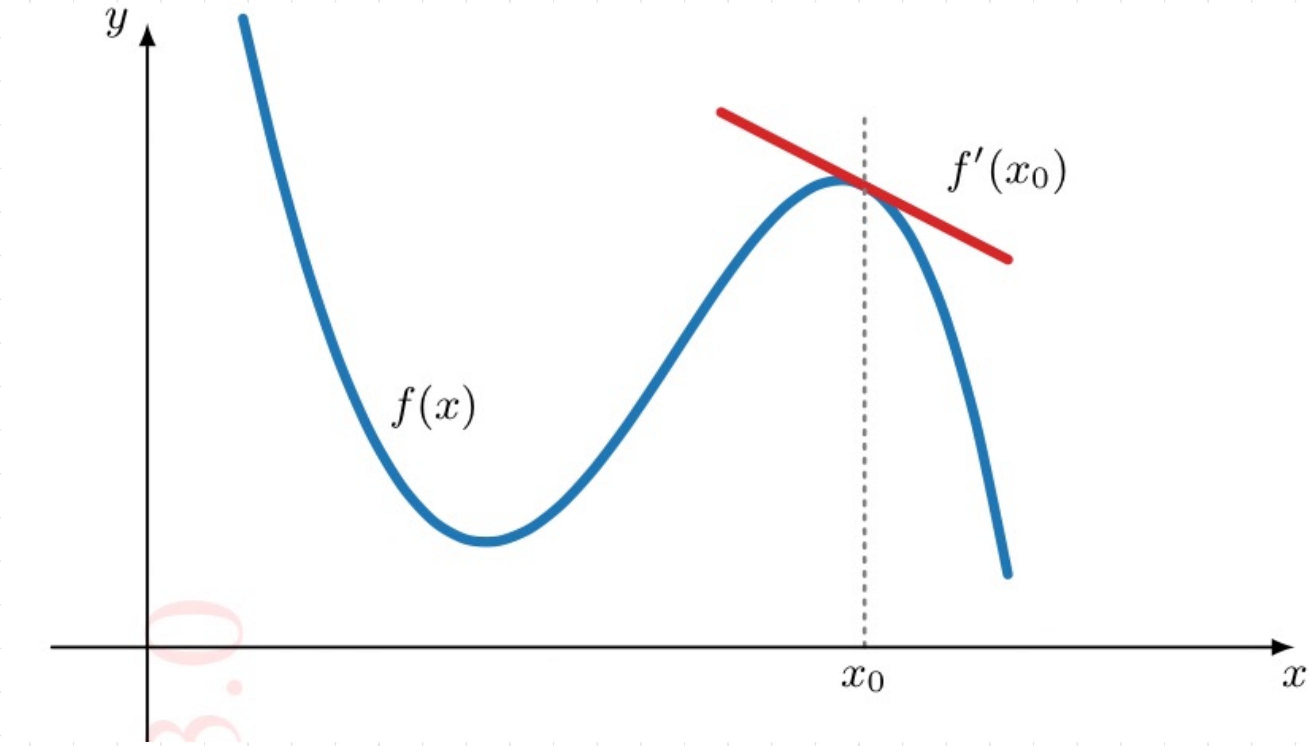
3. Say something about error.

$$\{(x_k, t_\ell) \mid x_k = kh_x,\ t_\ell = \ell h_t,\ \text{with } k, \ell \in \mathbb{Z} \text{ and } \ell \geq 0\} \quad \text{on } \mathbb{R} \times [0, \infty).$$



h_t also Δt or k

h_x also Δx or h



Back to basic derivatives

$$f(x_0 + h_x) = f(x_0) + f'(x_0)\,h_x + \frac{f''(\xi)\,h_x^2}{2}$$

$$f'(x_0) = \frac{f(x_0+h_x) - f(x_0)}{h_x} - \frac{f''(\xi)\,h_x}{2}$$

"fwd diff"

Similarly:

$$\boxed{f'(x_0) = \frac{f(x_0) - f(x-h_x)}{h_x}} - \frac{f''(\xi^-)h_x^2}{2}$$

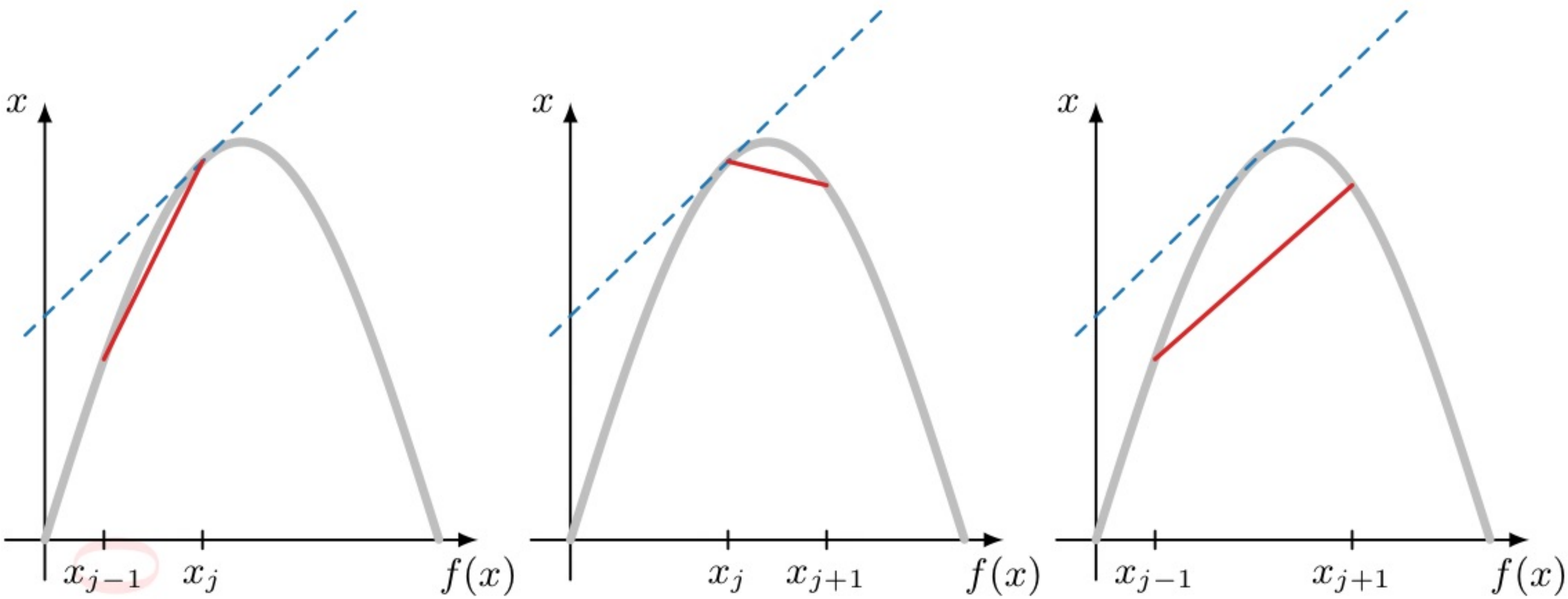
This is first-order

$$f(x_0+h_x) = f(x_0) + f'(x_0)h_x + \frac{f''(x_0)h_x^2}{2} + \frac{f'''(\xi^+)h_x^3}{6}$$

$$f(x_0-h_x) = f(x_0) - f'(x_0)h_x + \frac{f''(x_0)h_x^2}{2} - \frac{f'''(\xi^-)h_x^3}{6}$$

$$f(x_0+h_x) - f(x_0-h_x) = 2f'(x_0)h_x + \underline{h_x^3}\left(f'''(\xi^+) + f'''(\xi^-)\right)$$

$$f'(x_0) = \frac{f(x_0+h_x) - f(x_0-h_x)}{2h_x} - \frac{h_x^2}{12}\left(f'''(\xi^+) + f'''(\xi^-)\right)$$



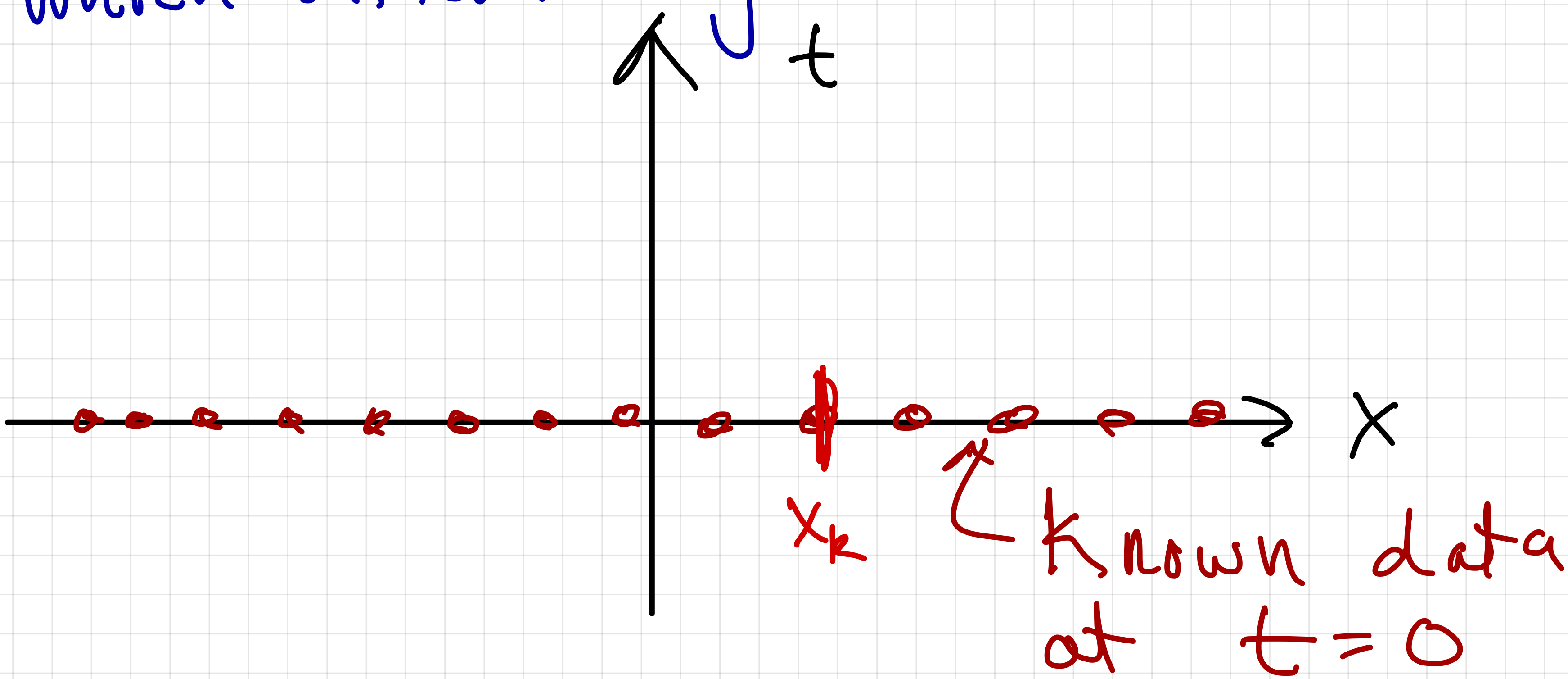
bkd | Fwd | Centered

Now consider

$$u_t + a u_x = 0 \qquad a > 0$$

$$\frac{d\, u(x,t)}{dt} + a \frac{d u(x,t)}{dx} = 0$$

Which differencing should we use?

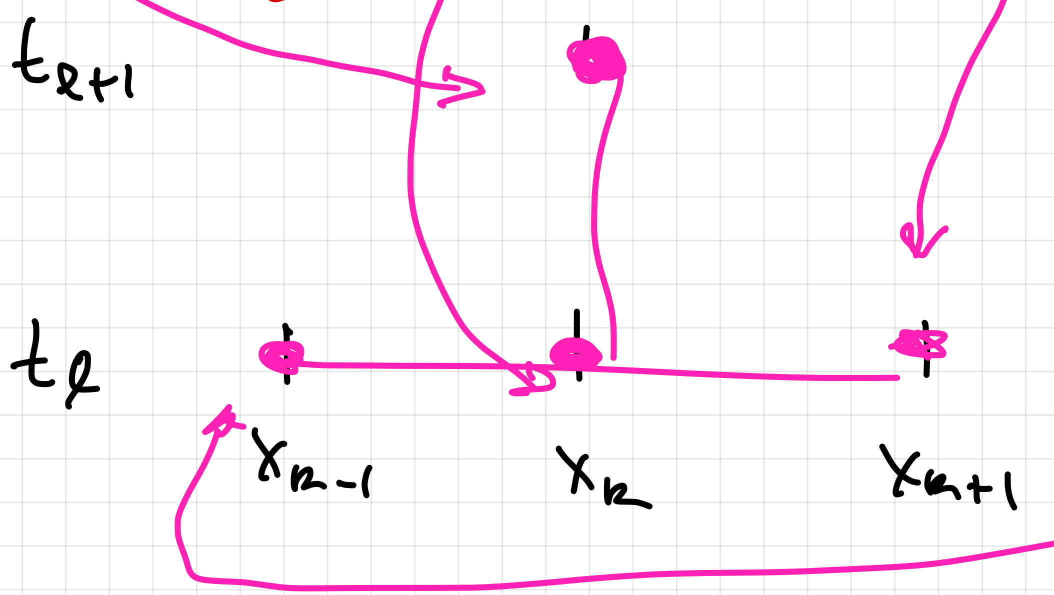


Try this

$$\underbrace{\frac{d\,u(x,t)}{dt}}_{\substack{\text{Forward time} \\ \text{at } x = x_k}} + a\,\underbrace{\frac{du(x,t)}{dx}}_{\substack{\text{centered diff in } x \\ \text{at } t = t_\ell}} = 0$$

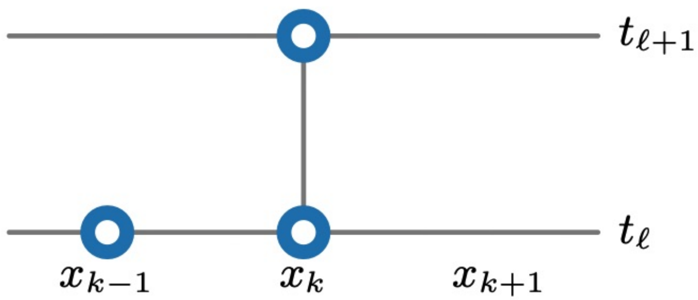
$$\frac{u(x_k, t_\ell + h_t) - u(x_k, t_\ell)}{h_t} + a\,\frac{u(x_k + h_x, t_\ell) - u(x_k - h_x, t_\ell)}{2h_x} = 0$$

$t_\ell + h_t = t_{\ell+1}$



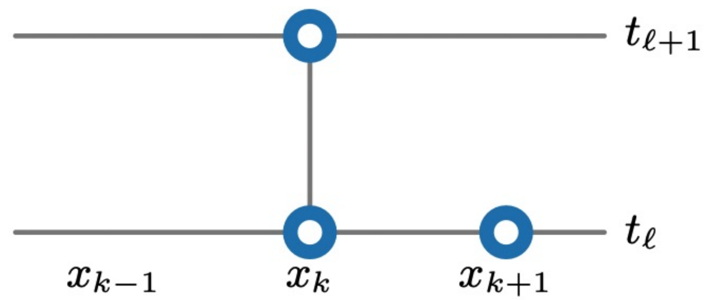
Many options!

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\frac{u_{k,\ell} - u_{k-1,\ell}}{h_x} = 0$$



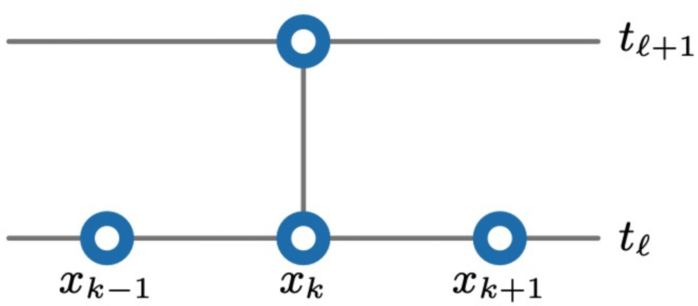
a. Explicit time, backward space (ETBS)

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\frac{u_{k+1,\ell} - u_{k,\ell}}{h_x} = 0$$



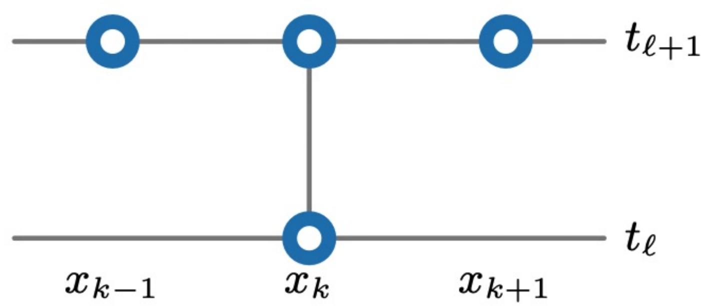
b. Explicit time, forward space (ETFS)

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\frac{u_{k+1,\ell} - u_{k-1,\ell}}{2h_x} = 0$$



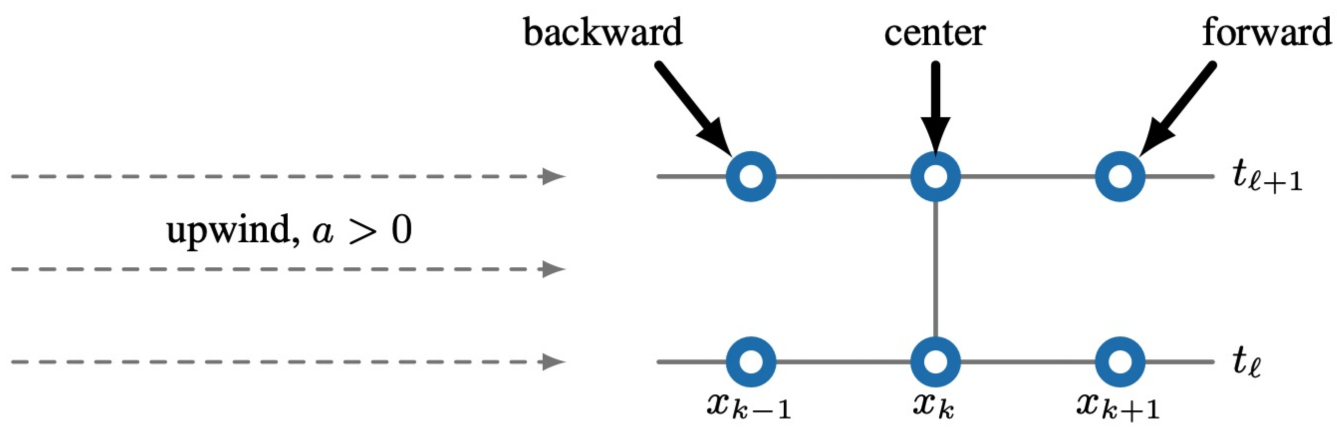
c. Explicit time, centered space (ETCS)

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\frac{u_{k+1,\ell+1} - u_{k-1,\ell+1}}{2h_x} = 0$$



d. Implicit time, centered space (ITCS)

A note on terminology:



<u>Explicit</u>: compute values at time $t_{\ell+1}$ using <u>only</u> values at time t_ℓ

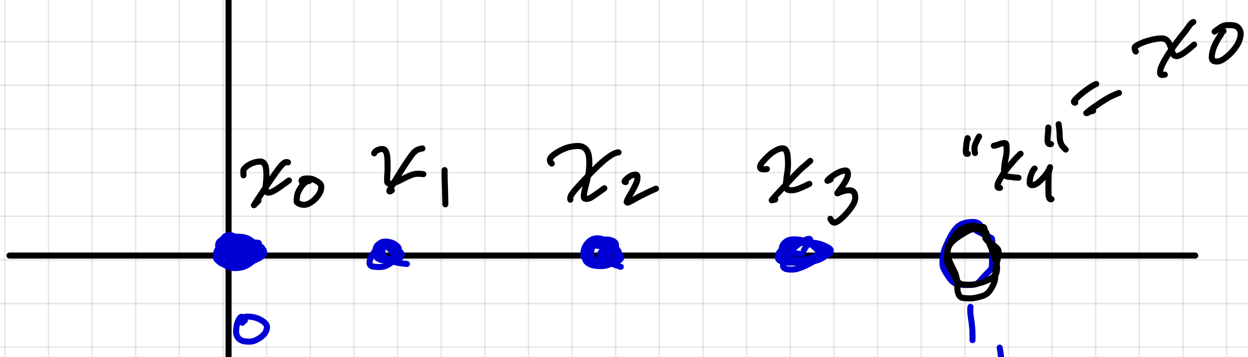
<u>Implicit</u>: compute values at time $t_{\ell+1}$ using other values at time $t_{\ell+1}$ (and t_ℓ)

Try it! **ETBS**

$$u_t + c u_x = 0 \quad \text{on } [0, 1]$$

let $c = 1$

periodic

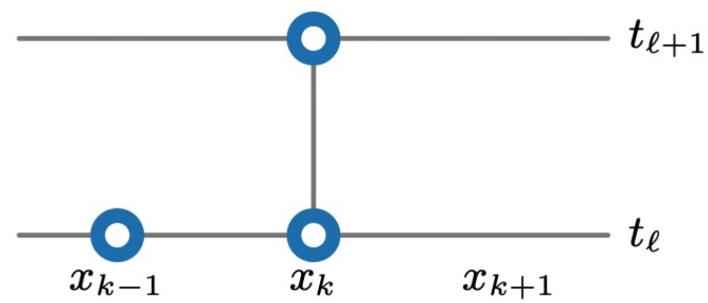


Start with $n_x = 50$

$\rightarrow h_x =$

Let $h_t =$

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\frac{u_{k,\ell} - u_{k-1,\ell}}{h_x} = 0$$



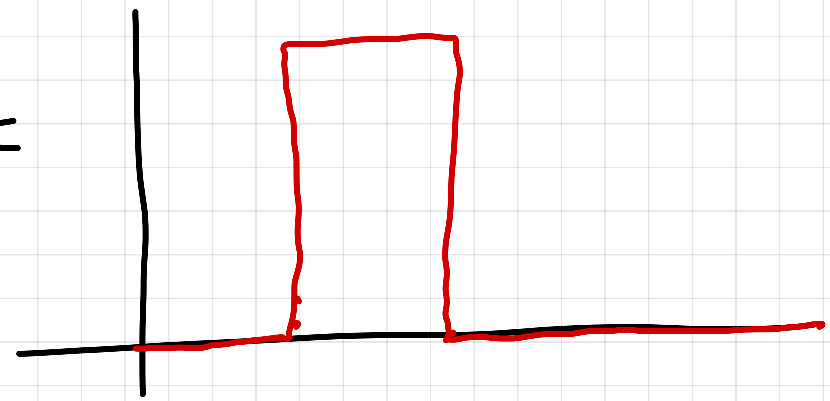
a. Explicit time, backward space (ETBS)

$$u_t + c\,u_x = 0$$

$$c = 1$$

on $[0,1]$, periodic

let $u(x,0) =$



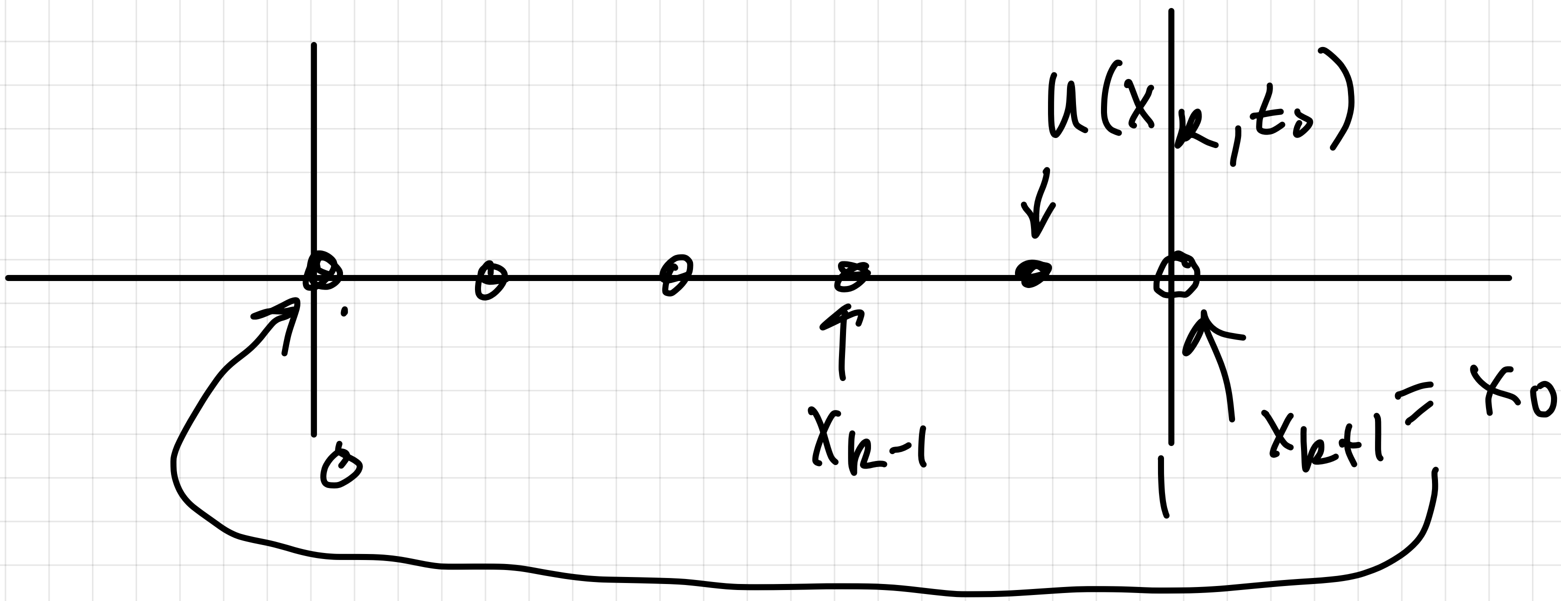
Steps

1. define a grid $x = $ np.linspace
2. define step size h_t
3. plot initial solution

$$\frac{u_{k\,l+1} - u_{kl}}{h_t} + c\,\frac{u_{kl} - u_{k-1\,l}}{h_x} = 0$$

$$u_{k\,l+1} - u_{kl} + \frac{c h_t}{h_x} \cdot (u_{kl} - u_{k-1\,l}) = 0$$

$$u_{k\,l+1} = \left(1 - \frac{c h_t}{h_x}\right) u_{kl} + \frac{c h_t}{h_x}\, u_{k-1\,l}$$



$$u_{k\,l+1} = \left(1 - \frac{c h_t}{h_x}\right) u_{k\,l} + \frac{c h_t}{h_x} u_{k-1\,l}$$

$$= u_{k\,l} - \frac{c h_t}{h_x} \cdot \left(u_{k\,l} - u_{k-1\,l}\right)$$

$$\lambda$$

$$= u_{k\,l} - \lambda \left(u_{k\,l} - u_{k-1\,l}\right)$$

u = f(x) # init condition

u = u[J] − λ (u[J] − u[Jm1])

Open questions

1. Why does "smooth" and get smaller?
2. Why does not travel at the correct speed?
3. Is it "accurate"?
4. Why does it "blow up" for $\lambda > 1$?

let $\underline{u}_\ell = \begin{bmatrix} \vdots \\ u_{-1,\ell} \\ u_{0,\ell} \\ u_{1,\ell} \\ \vdots \end{bmatrix}$

$\underline{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix}$

let exact solution be

$\underline{U}_\ell = \begin{bmatrix} \vdots \\ u(x_{-1}, t_\ell) \\ u(x_0, t_\ell) \\ \vdots \end{bmatrix}$

$\rightarrow \underline{U} = \begin{bmatrix} U_0 \\ U_1 \\ \vdots \end{bmatrix}$

$e_{k,\ell} = u(x_k, t_\ell) - u_{k,\ell}$

$\underline{e}_\ell = \underline{U}_\ell - \underline{u}_\ell$

$\underline{e} = \underline{U} - \underline{u}$

Definition 5.7: Two-Level Linear Finite-Difference Scheme

A finite-difference scheme that can be written as,

$$P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell + h_t \boldsymbol{b}_\ell, \tag{5.5}$$

is called a two-level linear finite-difference scheme. Each iteration depends only on two instances of time. Examples are given in Example 5.8.

ETBS: $$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\,\frac{u_{k,\ell} - u_{k-1,\ell}}{h_x} = 0$$

$$u_{k,\ell+1} - u_{k,\ell} + \frac{a h_t}{h_x}\left(u_{k,\ell} - u_{k-1,\ell}\right) = 0$$

$$u_{k,\ell+1} = \left(1 - \frac{a h_t}{h_x}\right) u_{k,\ell} + \frac{a h_t}{h_x} u_{k-1,\ell}$$

$$\Longrightarrow \quad P_h = I$$

$$Q_h = \operatorname{tridiag}\left(\frac{a h_t}{h_x},\ 1 - \frac{a h_t}{h_x},\ 0\right)$$

$u_{k+1,\ell}$

Definition 5.10: Truncation Error

The local truncation error, $\tau_{k,\ell}$, is the error that remains when a finite-difference method is applied to the exact solution, $u(x_k, t_\ell)$.

Example 5.12: ETFS Truncation Error

$$\begin{aligned}
\tau_{k,\ell} =& \frac{u(x_k, t_{\ell+1}) - u(x_k, t_\ell)}{h_t} + a\frac{u(x_{k+1}, t_\ell) - u(x_k, t_\ell)}{h_x} \\
=& \frac{1}{h_t}\left(u(x_k, t_\ell) + u_t(x_k, t_\ell)h_t + u_{tt}(x_k, \varsigma)\frac{h_t^2}{2} - u(x_k, t_\ell)\right) \\
&+ \frac{a}{h_x}\left(u(x_k, t_\ell) + u_x(x_k, t_\ell)h_x + u_{xx}(\xi^+, t_\ell)\frac{h_x^2}{2} - u(x_k, t_\ell)\right) \\
=& u_{tt}(x_k, \varsigma)\frac{h_t}{2} + a u_{xx}(\xi^+, t_\ell)\frac{h_x}{2} \\
=& \mathcal{O}(h_t, h_x)
\end{aligned}$$

Example 5.14: ETCS Truncation Error (Matrix Form)

For ETCS, $P_h = I$ and

$$\begin{aligned}
(P_h \boldsymbol{U}_{\ell+1})_k &= u(x_k, t_{\ell+1}) = u(kh_x, (\ell+1)h_t), \\
(Q_h \boldsymbol{U}_\ell)_k &= u(kh_x, \ell h_t) - \frac{ah_t}{2h_x}\Big(u((k+1)h_x, \ell h_t) - u((k-1)h_x, \ell h_t)\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
(P_h \boldsymbol{U}_{\ell+1} - Q_h \boldsymbol{U}_\ell)_k =& u_t(kh_x, \ell h_t)h_t + u_{tt}(kh_x, \varsigma)\frac{h_t^2}{2} + ah_t u_x(kh_x, \ell h_t) \\
&+ \frac{ah_t}{12}\Big(u_{xxx}(\xi^+, \ell h_t) + u_{xxx}(\xi^-, \ell h_t)\Big)h_x^2 \\
=& \Big(u_{tt}(kh_x, \varsigma)\frac{h_t}{2} + \frac{a}{12}(u_{xxx}(\xi^+, \ell h_t) + u_{xxx}(\xi^-, \ell h_t))h_x^2\Big)h_t \\
=& \tau_{k,\ell} h_t.
\end{aligned}$$

In general, $P_n U_{\ell+1} = Q_n U_\ell + b_\ell h_t + \tau_\ell h_t$

$$\tau_\ell = \begin{bmatrix} \tau_{-1\ell} \\ \tau_{0\ell} \\ \tau_{1\ell} \end{bmatrix}$$

$\underline{b=0}$

$$P_n U_{\ell+1} = Q_n U_\ell + \tau_\ell h_t$$

$$P_n u_{\ell+1} = Q_n u_\ell$$

$$\Rightarrow P_n e_{\ell+1} = Q_n u_\ell + \tau_\ell h_t$$

$$\Rightarrow e_{\ell+1} = P_n^{-1} Q_n u_\ell + P_n^{-1} \tau_n h_{t,}$$

- the max-norm (ℓ_∞): $\|\boldsymbol{e}\|_\infty = \max\limits_{k,\ell} |e_{k,\ell}|$; or

- the scaled Euclidean norm (ℓ_2): $\|\boldsymbol{e}\|_2 = \left(\sum\limits_k \sum\limits_\ell h_x h_t (e_{k,\ell})^2\right)^{\frac{1}{2}}$