Today 2/1

$$Z(x,t) = z_0 e^{i(kx - \omega t)} = z_0 e^{i2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)}$$

$z_0 =$ amplitude

$k =$ wave #

$\omega =$ ang frequency

$T =$ period

$= \frac{1}{\nu} \leftarrow$ frequency

$\omega = 2\pi\nu$

$\lambda =$ wave length

$= \frac{2\pi}{k}$

Main Tool

let $z = z_0 e^{i(kx - \omega t)}$

$$\boxed{Lz = \hat{\lambda} z}$$

for any linear operator L.

$L = \partial_t$: $\partial_t z = -i\omega z$

also

$L = \partial_x$: $\partial_x z = ik z$

$$\boxed{z \text{ solve } Lz = 0 \quad \text{iff } \lambda = 0}$$

$\rightarrow \lambda = 0$ is called the dispersion relation.

Example

$$L = \partial_t + a\,\partial_x$$

$$\rightarrow L z = -i\omega z + a i k z$$

$$= (-i\omega + a i k)\, z$$

$$= \underbrace{i\,(ak - \omega)}_{\lambda(k,\omega)}\, z$$

dispersion relation:

$$\lambda = 0$$

$$\Rightarrow i(ak - \omega) = 0$$

$$\Rightarrow \omega = ak$$

Example

$$L = \cancel{u_t + a u_x - D u_{xx}}$$
$$= \partial_t + a\partial_x - D\partial_{xx}$$

$\rightarrow$ $Lz = -i\omega z + aikz - D(ik)^2 z$
$$= (-i\omega + aik + Dk^2) z$$

$\rightarrow$ $\boxed{\omega = ak - iDk^2}$

take $z = z_0 e^{i(kx - \omega t)}$

let $\omega(k) = \underbrace{\alpha(k)}_{\text{real}} + i \underbrace{\beta(k)}_{\text{imag}}$

$$= z_0 e^{i(kx - \alpha t - i\beta t)}$$

$$= z e^{\beta t} e^{i(kx - \alpha t)}$$

dissipitive

if $\alpha(k) = ak$

then $e^{i(kx - \alpha t)}$ is a fixed wave at a fixed velocity

$$\Rightarrow v_{ph} = \frac{Re(\omega(k))}{k} = \frac{\alpha(k)}{k}$$

if v_{ph} is constant, then no dispersion

if v_{ph} is non-constant, then dispersive.

$$\rightarrow \quad z_{j,\ell} = z_0 e^{i(k\underset{\substack{\uparrow\\ jh_x}}{x} - \omega \underset{\substack{\uparrow\\ \ell h_t}}{t})}$$

$$= z_0 e^{i(kjh_x - \omega \ell h_t)}$$

Example FTBS

$$u_{j\ell+1} = \gamma u_{j-1,\ell} + (1-\gamma) u_{j\ell}$$

let $u = z$:

$$\cancel{z_{j\ell}} e^{-i\omega h_t} = \gamma \cancel{z_{j\ell}} e^{-ikh_x} + (1-\gamma) \cancel{z_{j\ell}}$$

$$\Rightarrow \quad e^{-i\omega h_t} = \underbrace{\gamma e^{-ikh_x} + (1-\gamma)}_{\text{amp factor}}$$

$$e^{-i\omega h_t} = \underbrace{\gamma e^{-ikh_x} + 1 - \gamma}$$

$s(kh_x)$ = symbol
or
amp factor

stability: if $|s(kh_x)| \leq 1$
then stable.

$kh_x = ?$ $\Rightarrow$ $\dfrac{2\pi h_x}{\lambda}$

$\omega = \omega(kh_x)$

look at s = complex

$$= |s|\, e^{i\phi} \quad (\text{some } \phi)$$

$$= e^{\ln|s| + i\phi}$$

want $e^{-i\omega h t} = e^{\ln|s| + i\phi}$

goal: find ω

$$-i\omega h t = \ln|s| + i\phi$$

$$\omega h t = i\ln|s| - \phi$$

$$\omega = \frac{i\ln|s| - \phi}{ht}$$

back to z_{je}

$$z_{je} = z_0 \, e^{i(k j h_x - \omega \ell h_t)}$$

put it here

$$= z_0 \, e^{i\left(k j h_x - \frac{i \ln|s| - \phi}{h_t} \cdot \ell h_t\right)}$$

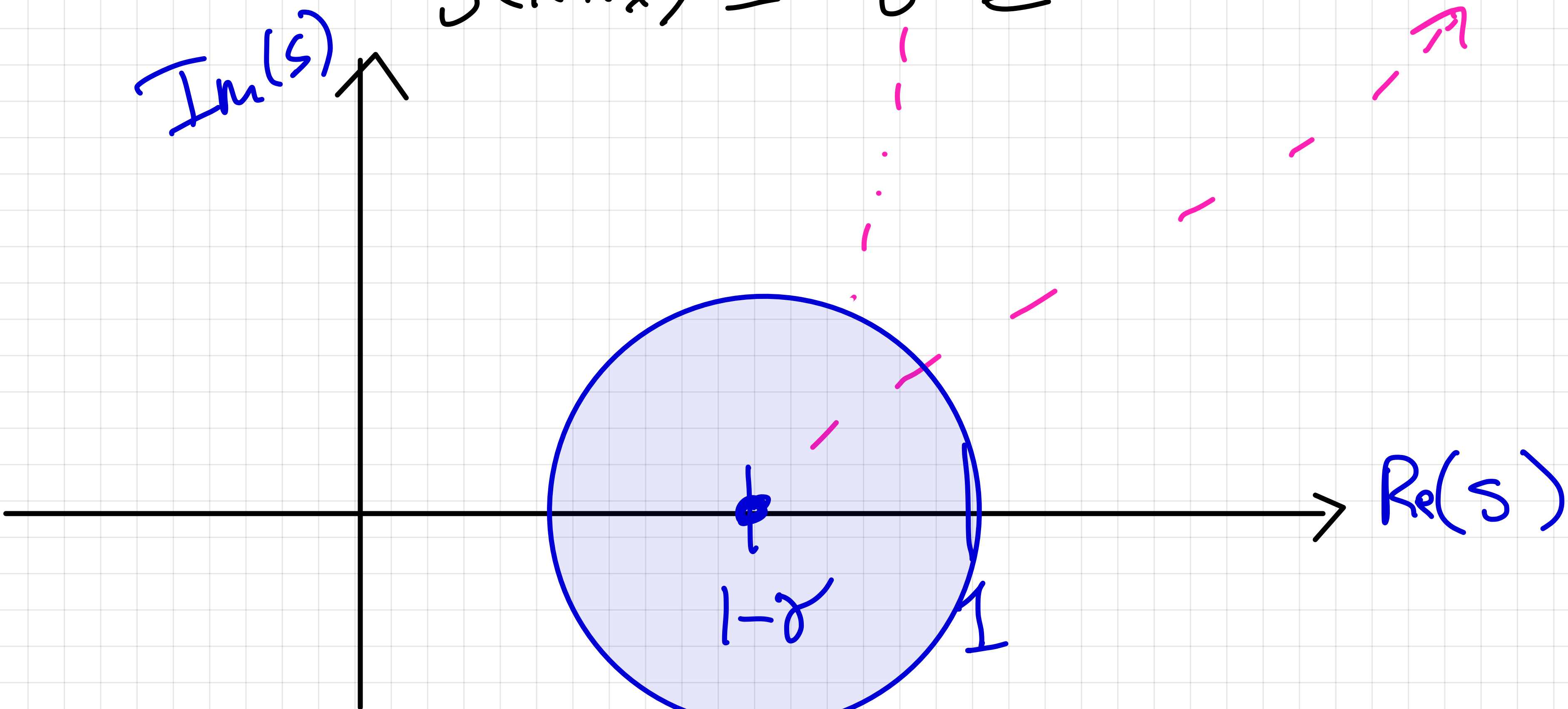
$$= z_0 \, e^{\ln|s| \cdot \ell} \, e^{i\left(k j h_x - \frac{-\phi}{h_t} \cdot \ell h_t\right)}$$

numerical phase velocity: $V_{ph} = - \frac{\phi(k h_x)}{k h_t}$

if not constant in k, then dispersive.

ETBS $\gamma = a \frac{h_t}{h_x}$

$S(kh_x) = \gamma e^{-ikh_x} + 1 - \gamma$



Dispersion? ETBS

let kh_x be small

$$\Rightarrow e^{-ikh_x} \approx 1 - ikh_x$$

$$\Rightarrow s(kh_x) = 1 - \gamma + \gamma e^{-ikh_x}$$

$$\approx 1 - \gamma + \gamma(1 - ikh_x)$$

$$= 1 - i\gamma kh_x$$

Also $e^{-i\omega h_t} \approx 1 - i\omega h_t$

$$\Rightarrow \omega h_t \approx \gamma k h_x$$

$$\Rightarrow \omega \approx \frac{\gamma k h_x}{h_t} = a \frac{\cancel{h_t}}{\cancel{h_x}} \cdot \frac{k \cancel{h_x}}{h_t}$$

$$\approx ak$$

Conservation Laws

1D: $\partial_t u + (f(u))_x = 0$

ND: $\partial_t u + \nabla \cdot (\underline{f}(u)) = 0$