Today 1/6 Section 6.1 - 6.2

- Conservation laws
- Shock + speeds
- Riemann Problem.

1D

Conservation law is of the form

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

If f is differentiable

Then $u_t + f'(u)\, u_x = 0$

2D $\frac{\partial u}{\partial t} + \nabla \cdot (\underline{f}(u)) = 0$

$$u_t + y u_x - x u_y = 0$$

$$u_t + \nabla \cdot (f(u)) = 0$$

$$\boxed{\nabla \cdot (\underline{b}\, u) = (\nabla \cdot b) u + \underline{b} \cdot \nabla u}$$

$$\Rightarrow u_t + \underbrace{(y, -x)^T}_{\underset{\sim}{b}} \cdot \nabla u = 0$$

$$\rightarrow u_t + \nabla \cdot \left((y, -x)^T u \right) = 0$$

Try: $f(u) = xu$

Then $\dfrac{\partial f(u)}{\partial x} = u + xu'$

Consider a space-time curve $x(t)$ that satisfies

$$\begin{cases} x'(t) = u(x(t), t) \\ x(0) = u(x(0), 0) \end{cases}$$

Then

$$\frac{d}{dt} u(x(t), t) = \frac{\partial u(x(t), t)}{\partial t} + \frac{dx(t)}{dt} \frac{\partial (u(x,t))}{\partial x}$$

$$= u_t + x'(t) u_x$$

Then

$$= 0$$

if $u_t + (u^2/2)_x = 0$

For $u_t + \left(\frac{u^2}{2}\right)_x = 0$

$x(t)$ are straight lines

given by $\frac{dx}{dt} = u \leftarrow$ initial condition

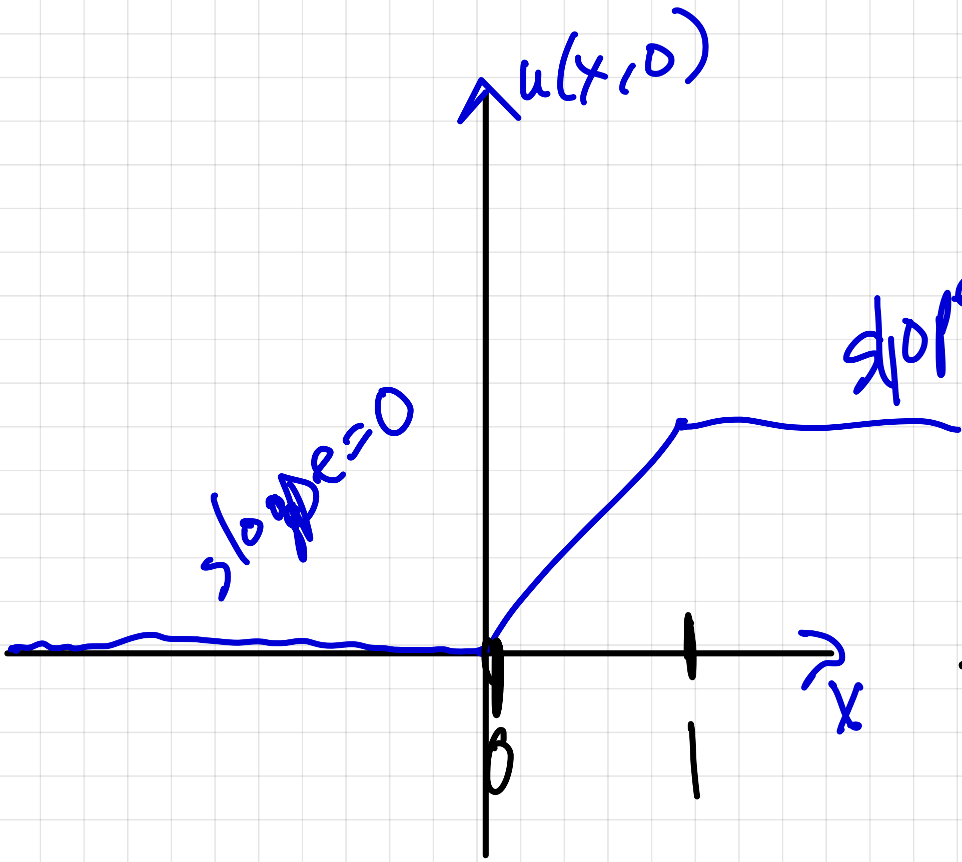
① $u_t + \left(\frac{u^2}{2}\right)_x = 0$

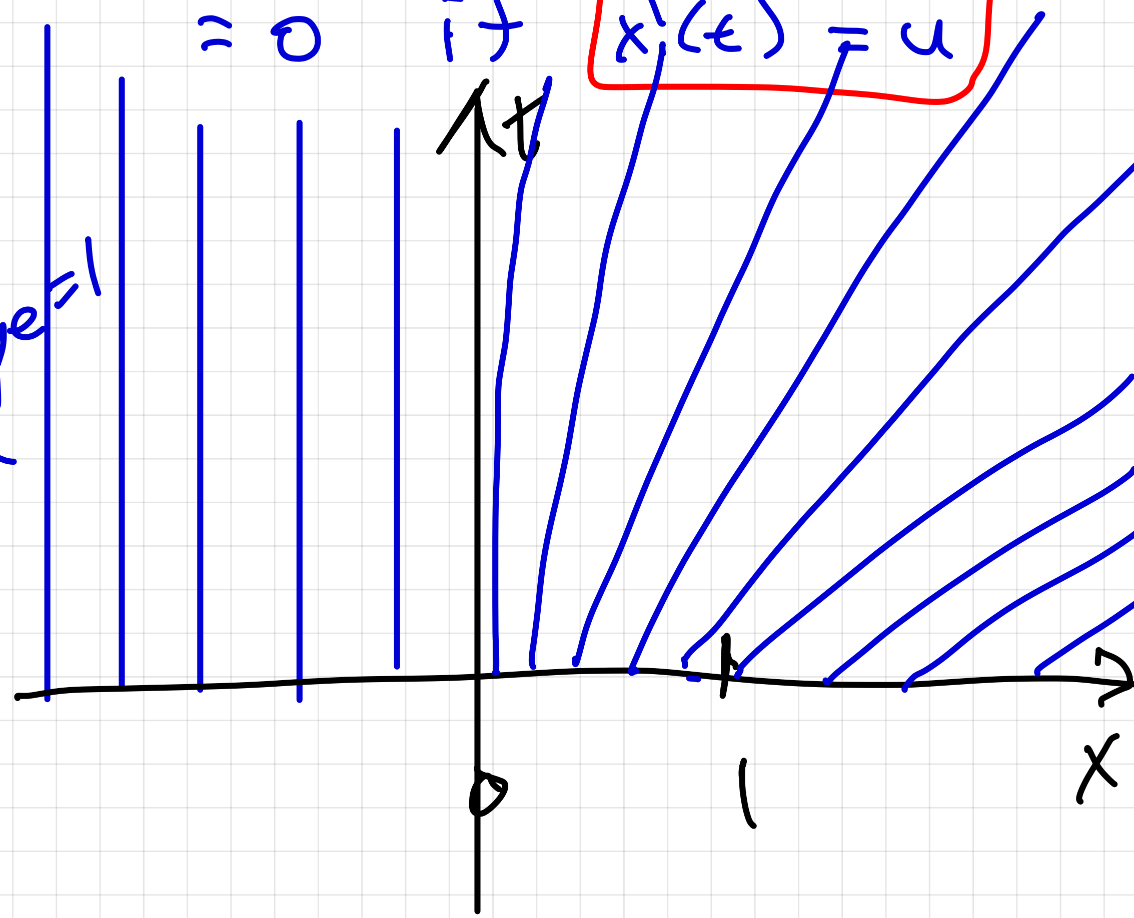
or

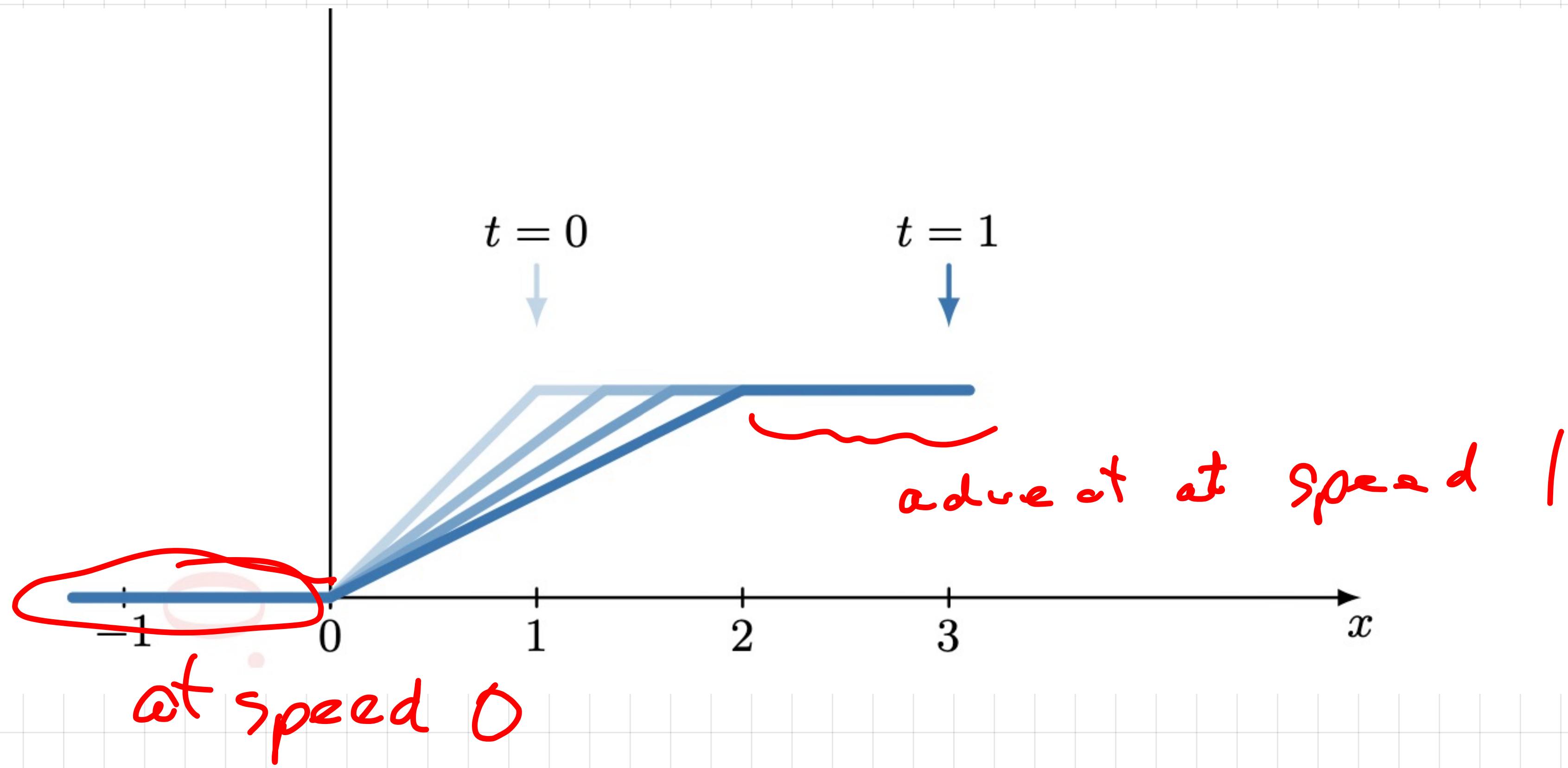
$u_t + u\,u_x = 0$

take $\frac{d\,u(x(t),t)}{dt} = \frac{\partial u}{\partial t} + x'(t)\frac{\partial u}{\partial x}$

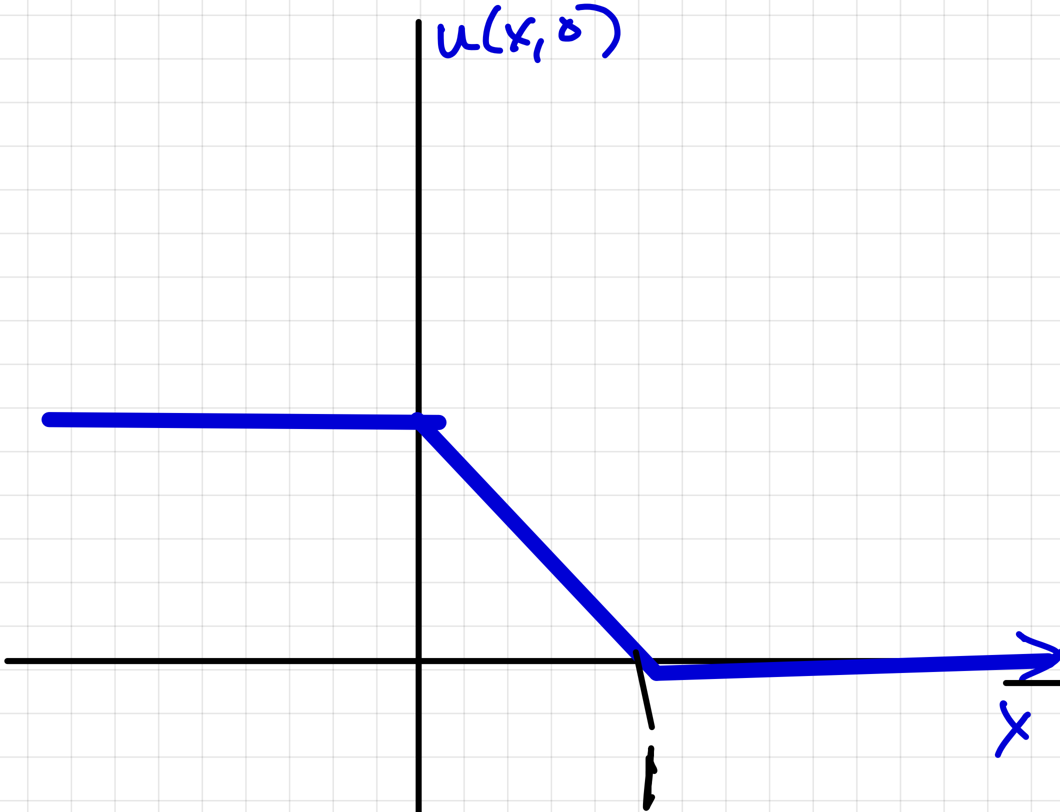
$= 0$ if $\boxed{x'(t) = u}$

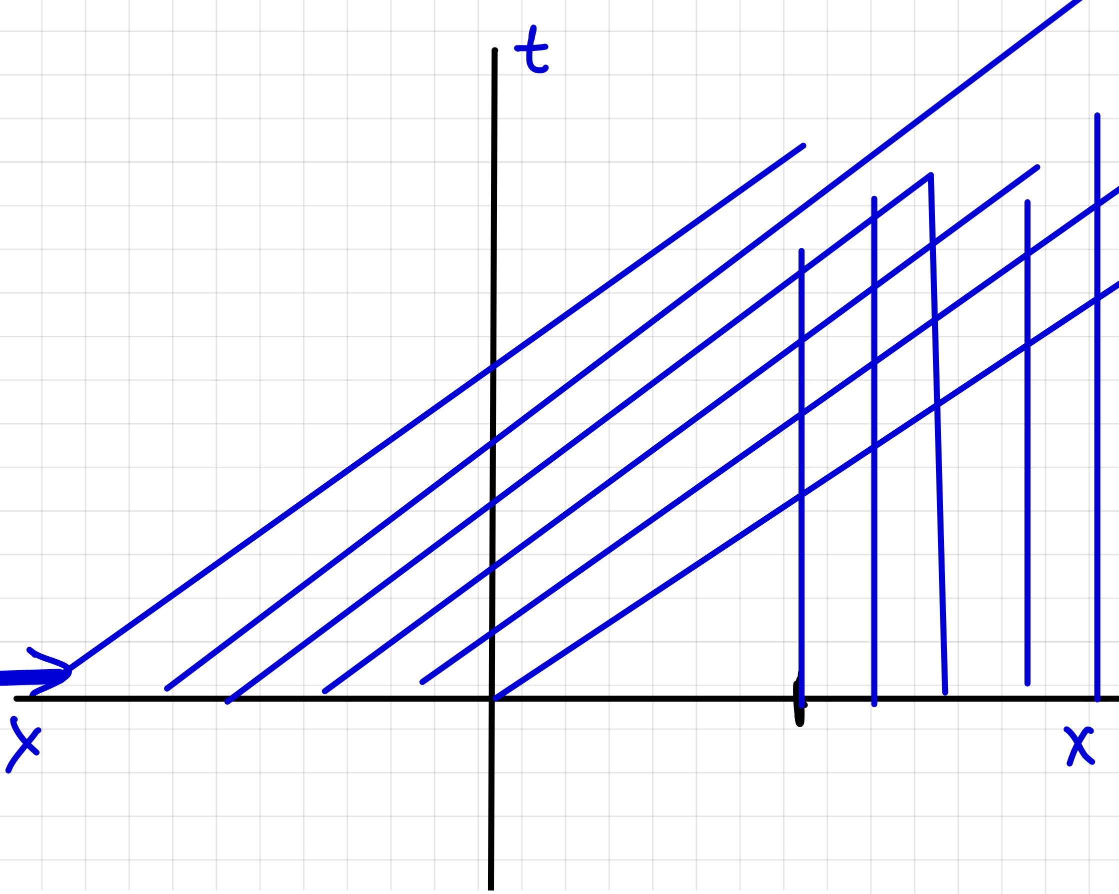


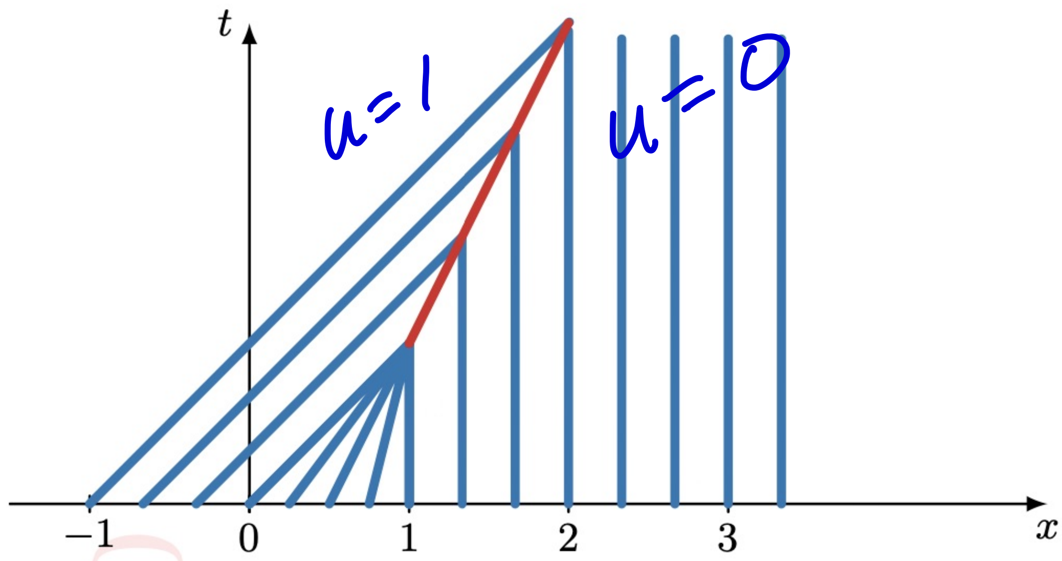


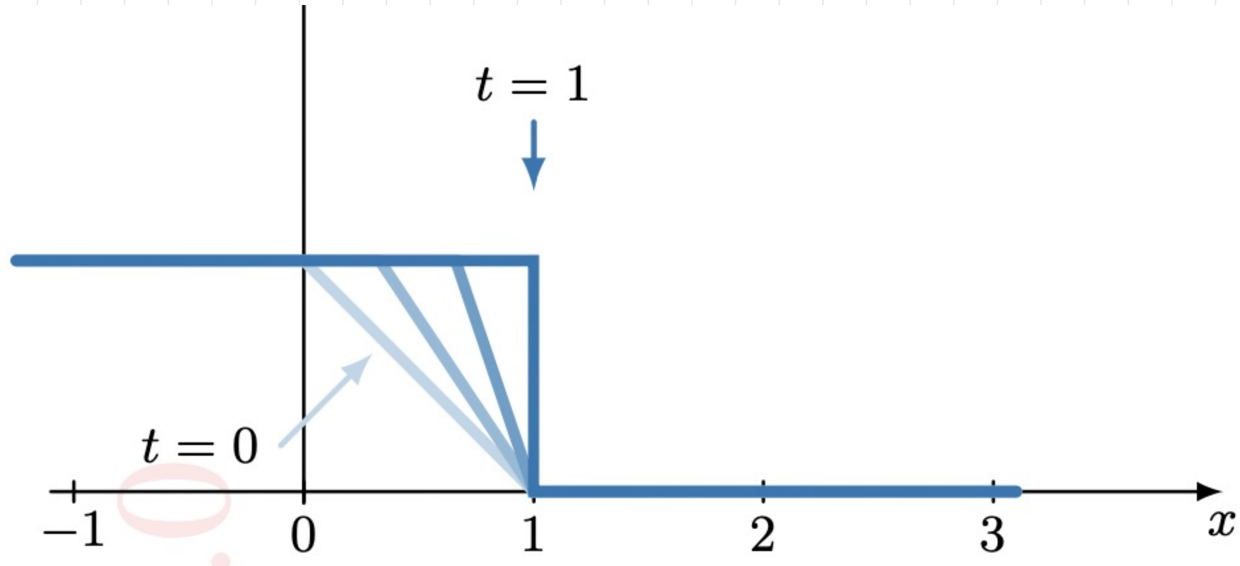


t = 0
t = 1
−1
0
1
2
3
x
advect at speed 1
at speed 0


u(x,0)
1
x


t
x

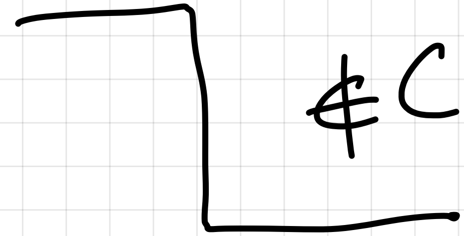

t
u=1
u=0
−1
0
1
2
3
x



t = 1
t = 0
−1
0
1
2
3
x

How to handle shocks?

... not in C^1



$\notin C^1$

Weak Solution V1.0:

u is a weak solution to

$$u_t + (f(u))_x = 0$$

if

$$\frac{d}{dt}\left(\int_a^b u(x,t)\,dx\right) + f(u(b,t)) - f(u(a,t)) = 0$$

$\forall\, a, b, t$ almost,

V2.0 consider $\phi \in C_0^1(\mathbb{R}^2)$

cont. ✓

differentiable ✓

zero outside some bounded set in $\mathbb{R}^2$

$$u_t + (f(u))_x = 0$$

$$\to \quad u_t \phi + (f(u))_x \phi = 0$$

$$\int_0^{\infty}\int_{-\infty}^{\infty} u_t \phi + (f(u))_x \phi \, dx\, dt = 0$$

$$\int_0^\infty \int_{-\infty}^\infty u_t \phi + (f(u))_x \phi \, dx\, dt = 0$$

IBP in "t" and "x":

$$-\int_0^\infty \int_{-\infty}^\infty u \, \phi_t \, dx\, dt + \int_{-\infty}^\infty u \phi \Big|_0^{\cancel{\infty}} dx$$

$$-\int_0^\infty \int_{-\infty}^\infty f(u) \cdot \phi_x \, dx\, dt + \int_0^\infty f(u)\phi \Big|_{\cancel{-\infty}}^{\cancel{\infty}} dt$$

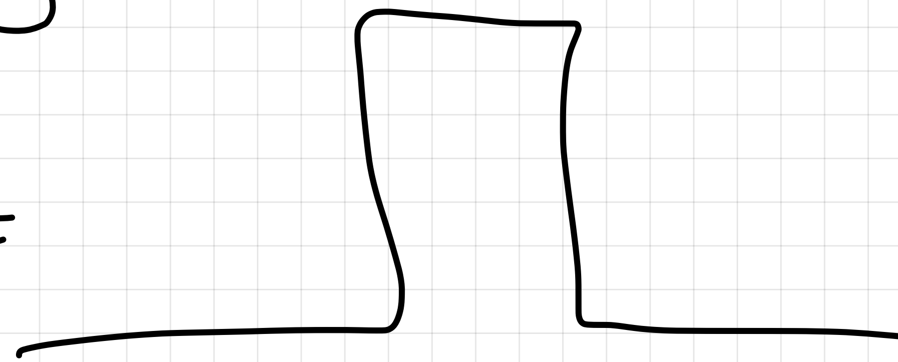
$$= 0$$

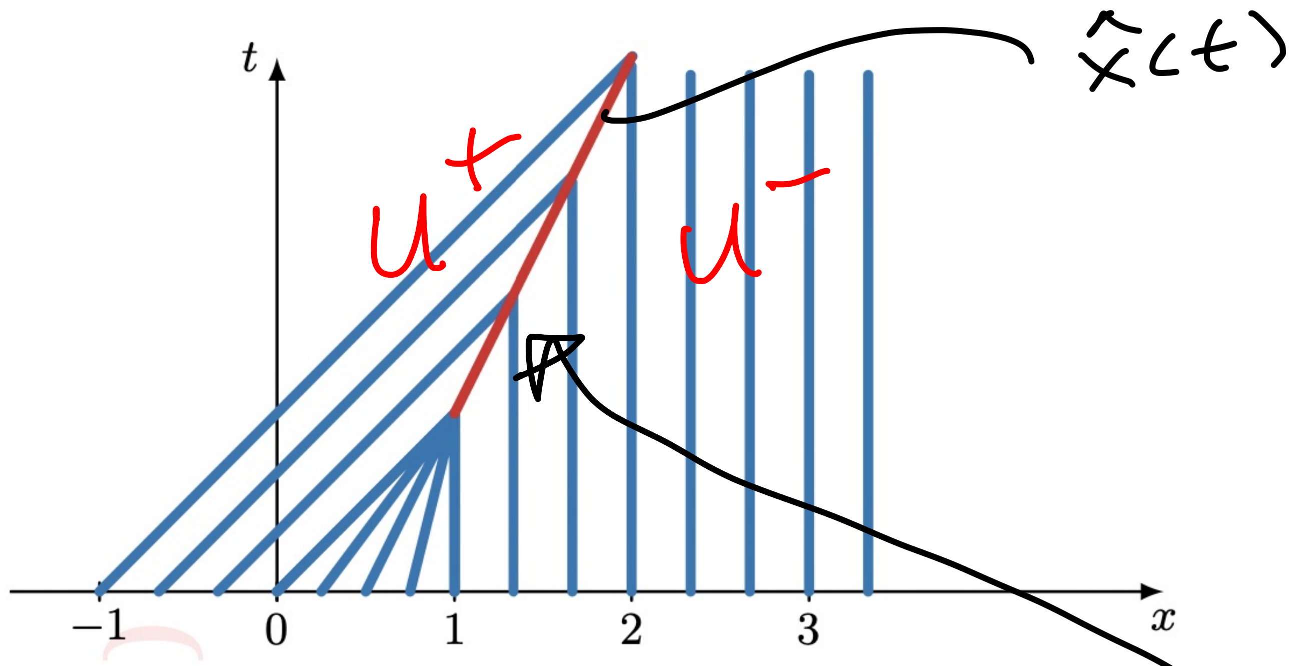
$$\Rightarrow \int_0^\infty \int_{-\infty}^\infty u \, \phi_t + f(u) \phi_x \, dx\, dt + \int_{-\infty}^\infty u(x,0)\, \phi(x,0)\, dx$$

$$= 0$$

$$u_t + u\, u_x = 0$$

$$u(x,0) =$$





let $\hat{x}(t)$ be the curve

Then $\hat{x}'(t) = \dfrac{f(u^+) - f(u^-)}{u^+ - u^-}$

$$f(u) = \frac{u^2}{2}$$

$$s = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

$$= \frac{\frac{1}{2}\cancel{(u^+ - u^-)}(u^+ + u^-)}{\cancel{u^+ - u^-}}$$

$$= \frac{u^+ + u^-}{2}$$