Today 2/8

FV Methods

Recap

① The Riemann Problem for

$$u_t + (f(u))_x = 0$$

is

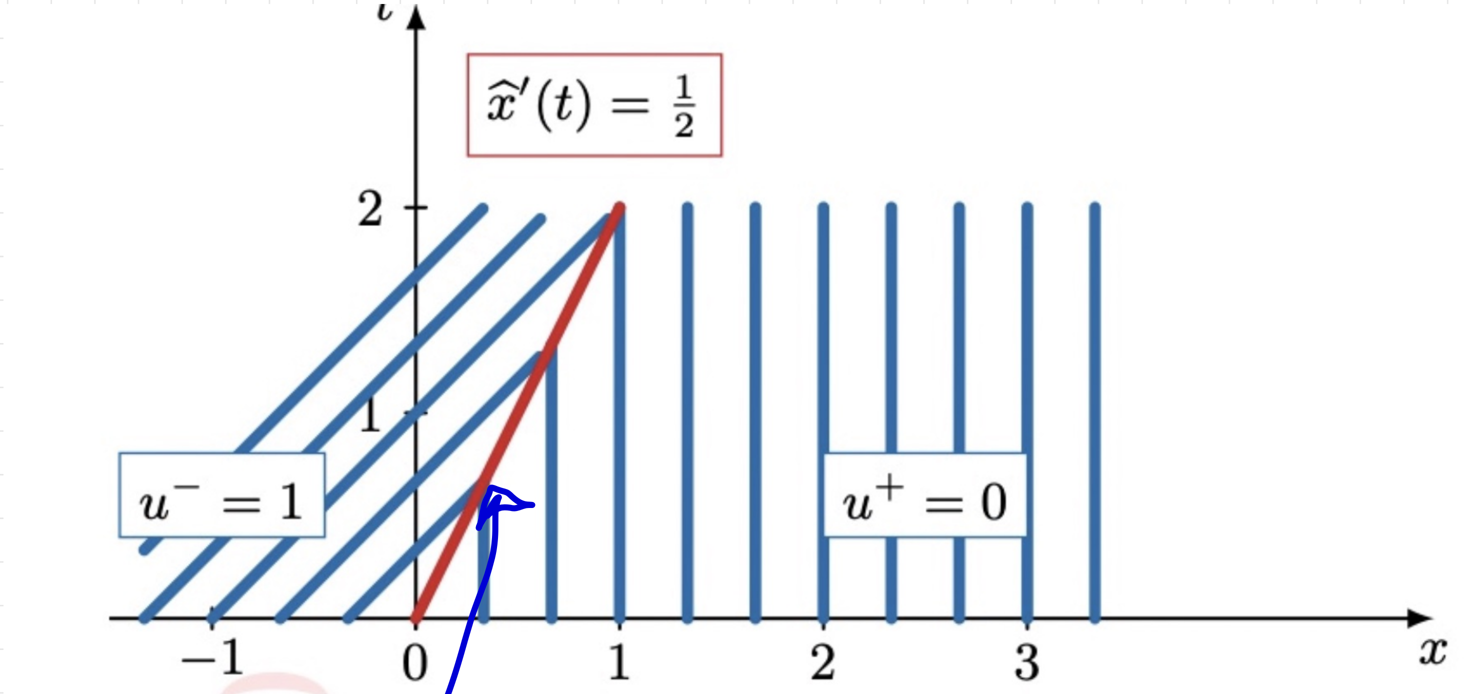
$$u(x,0) = \begin{cases} u^- & x \leq 0 \\ u^+ & x > 0 \end{cases}$$

② Two cases:

$u^- < u^+$: rarefaction

$u^- > u^+$: shock

③ Shocks:



$$\text{Speed} = \frac{f(u^-) - f(u^+)}{u^- - u^+}$$

$$\overset{\text{Burgers}}{=} \frac{u^- + u^+}{2}$$

Rarefaction

Both are weak solutions:

$$\int_0^\infty \int_{-\infty}^\infty u\,\phi_t + f(u)\,\phi_x \,dx\,dt + \int_{-\infty}^\infty u(x,0)\,\phi(x,0)\,dx = 0$$

Two tools:

① Vanishing viscosity:

$$u_t + \left(\frac{u^2}{2}\right)_x = \nu u_{xx}$$

\nu u

what happens when $\nu \to 0$?

② Entropy:

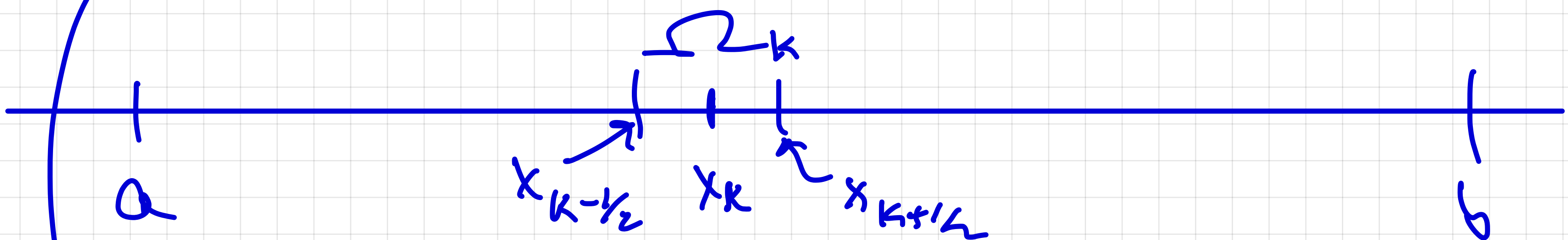
if flux f is convex $(f'' > 0)$

then $\hat{x}(t)$ satisfies the Lax entropy condition if

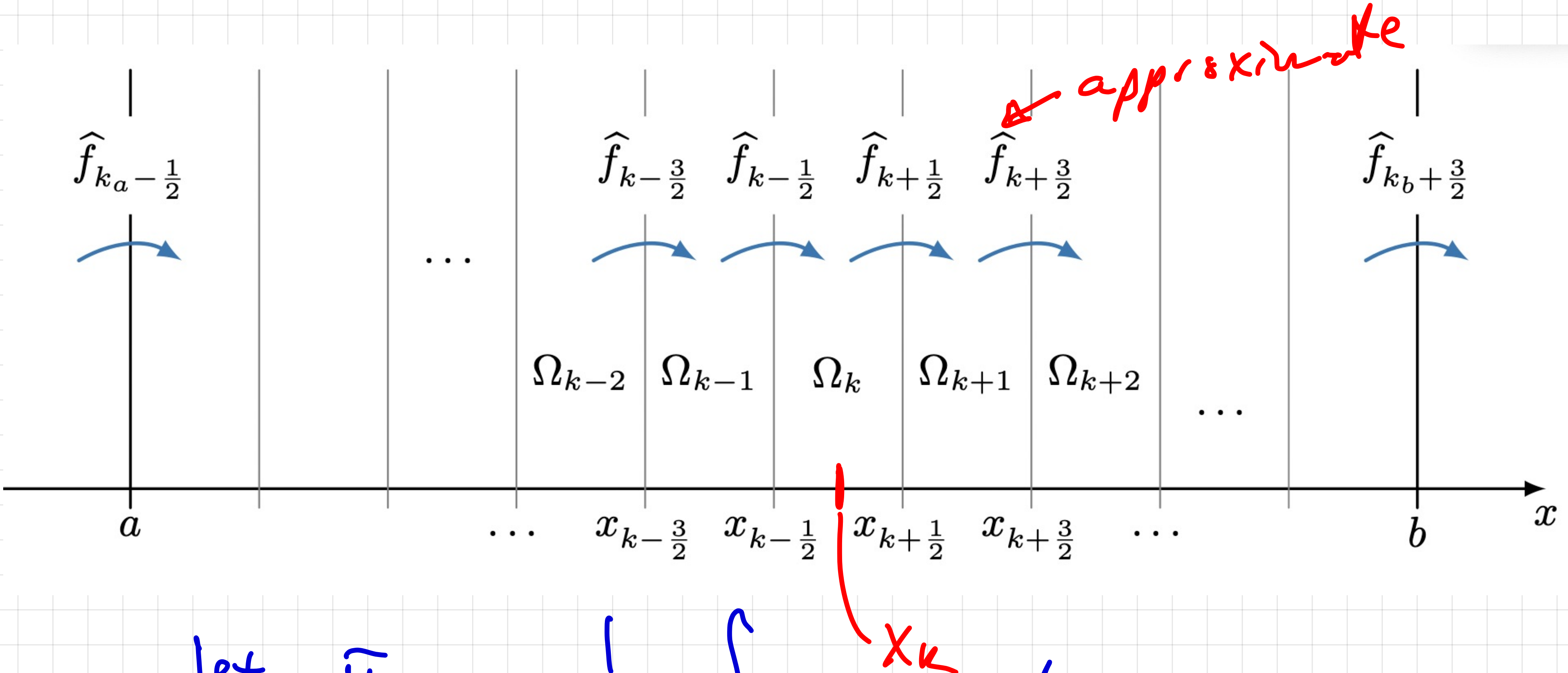
$$f'(u^-) > \hat{x}' > f'(u^+) \quad \forall t$$

Back to conservation:

$$\frac{d}{dt}\int_a^b u(x,t)\,dx + f(u(b,t)) - f(u(a,t)) = 0$$



$$\frac{d}{dt}\int_{\Omega_k} u(x,t)\,dx + f(u(x_{k+1/2},t)) - f(u(x_{k-1/2},t)) = 0$$



let $\bar{u}_k = \frac{1}{h_k} \int_{\Omega_k} u(x,t)\, dx$

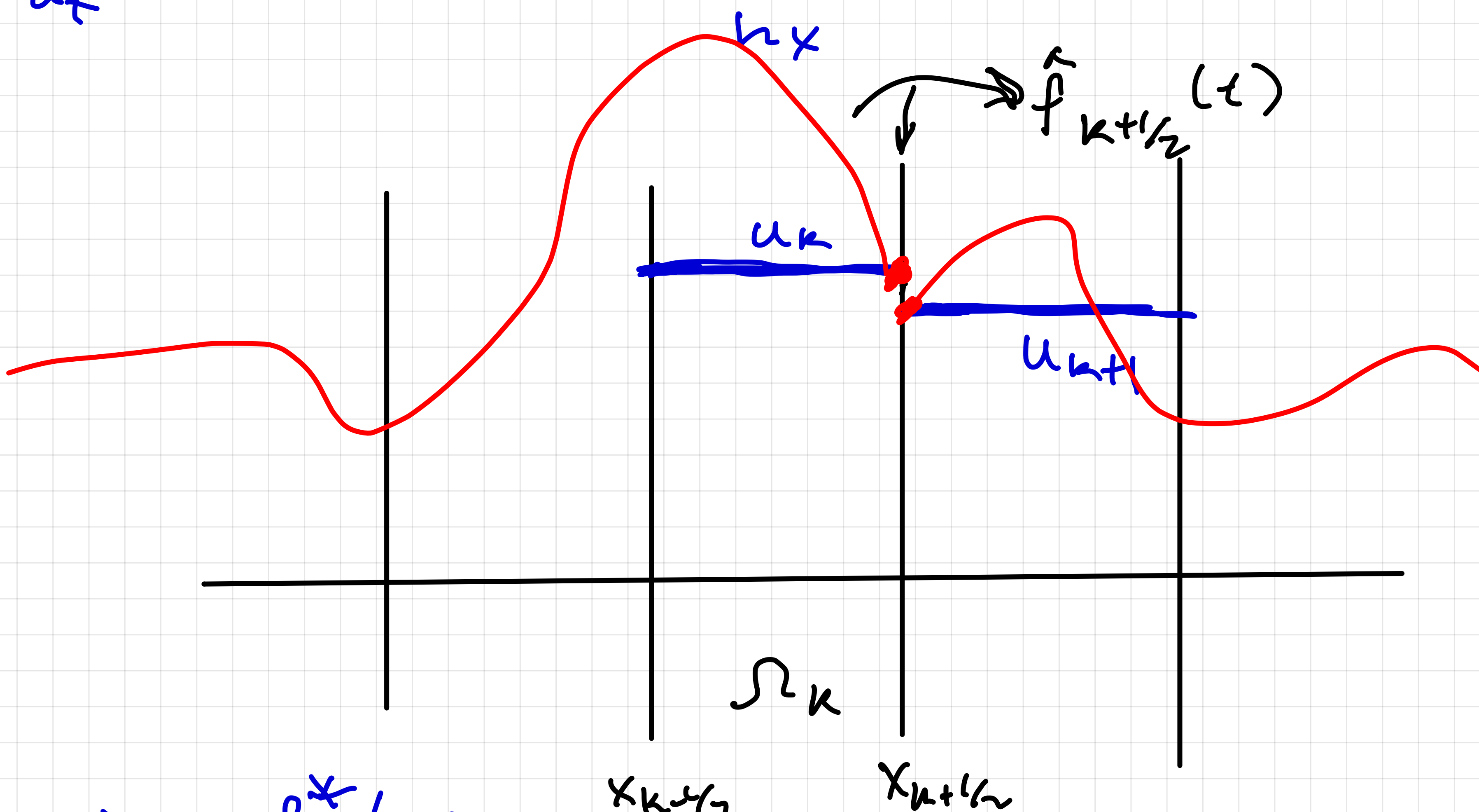
$$\frac{d}{dt}\bar{u}_k(t) + \frac{f(u(x_{k+1/2},t)) - f(u(x_{k-1/2},t))}{h_x} = 0$$

exact

let $u_k \approx \bar{u}_k$

let $\hat{f}_{k+1/2}(t) \approx f(u(x_{k+1/2},t))$

$$\frac{d}{dt} u_k(t) + \underline{\hat{f}_{k+1/2}(t) - \hat{f}_{k-1/2}(t)} = 0$$



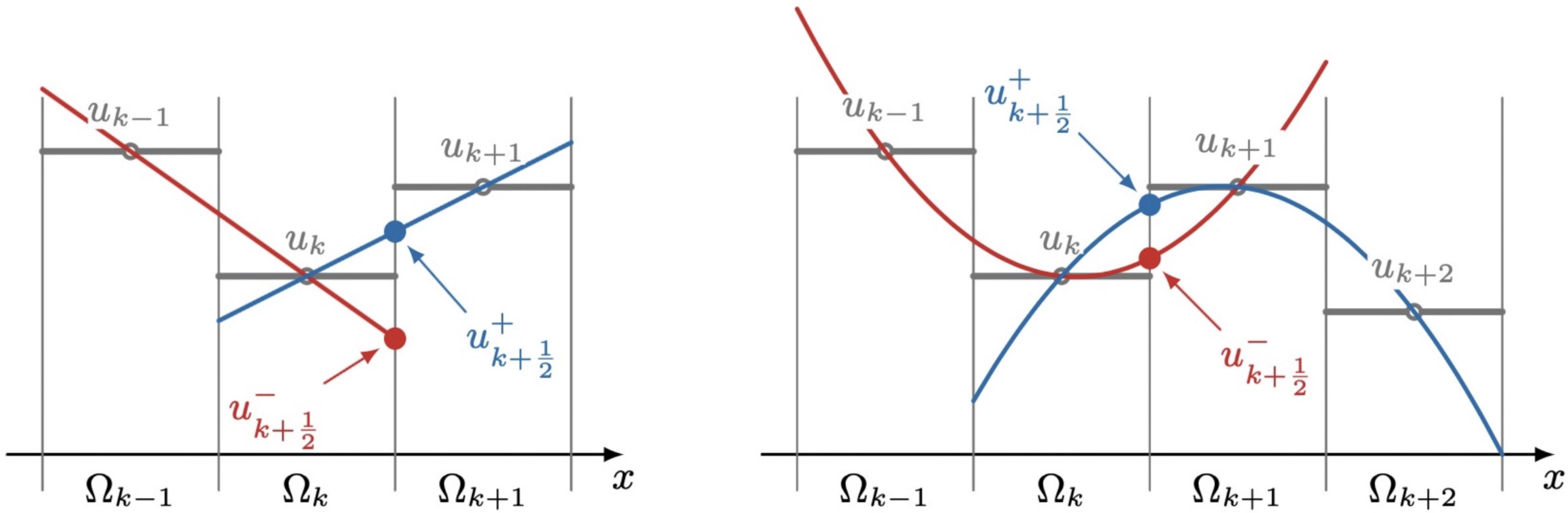
$$\hat{f}_{k+1/2}(t) = f^*\left(u^-_{k+1/2},\, u^+_{k+1/2}\right)$$

Easy case

take $u^-_{k+1/2} = u_k$

$u^+_{k+1/2} = u_{k+1}$

Better: linears or quadratics



Base method: $u_{k,\ell} \simeq \bar{u}_k(t_\ell)$

Explicit in time:

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + \frac{f^*(u_{k,\ell}, u_{k+1,\ell}) - f^*(u_{k-1,\ell}, u_{k,\ell})}{h_x} = 0$$

(Q1) What is the numerical flux for ETBS for $u_t + a u_x = 0$?

$a > 0$ $\quad \dfrac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + \dfrac{a u_{k,\ell} - a u_{k-1,\ell}}{h_x} = 0$

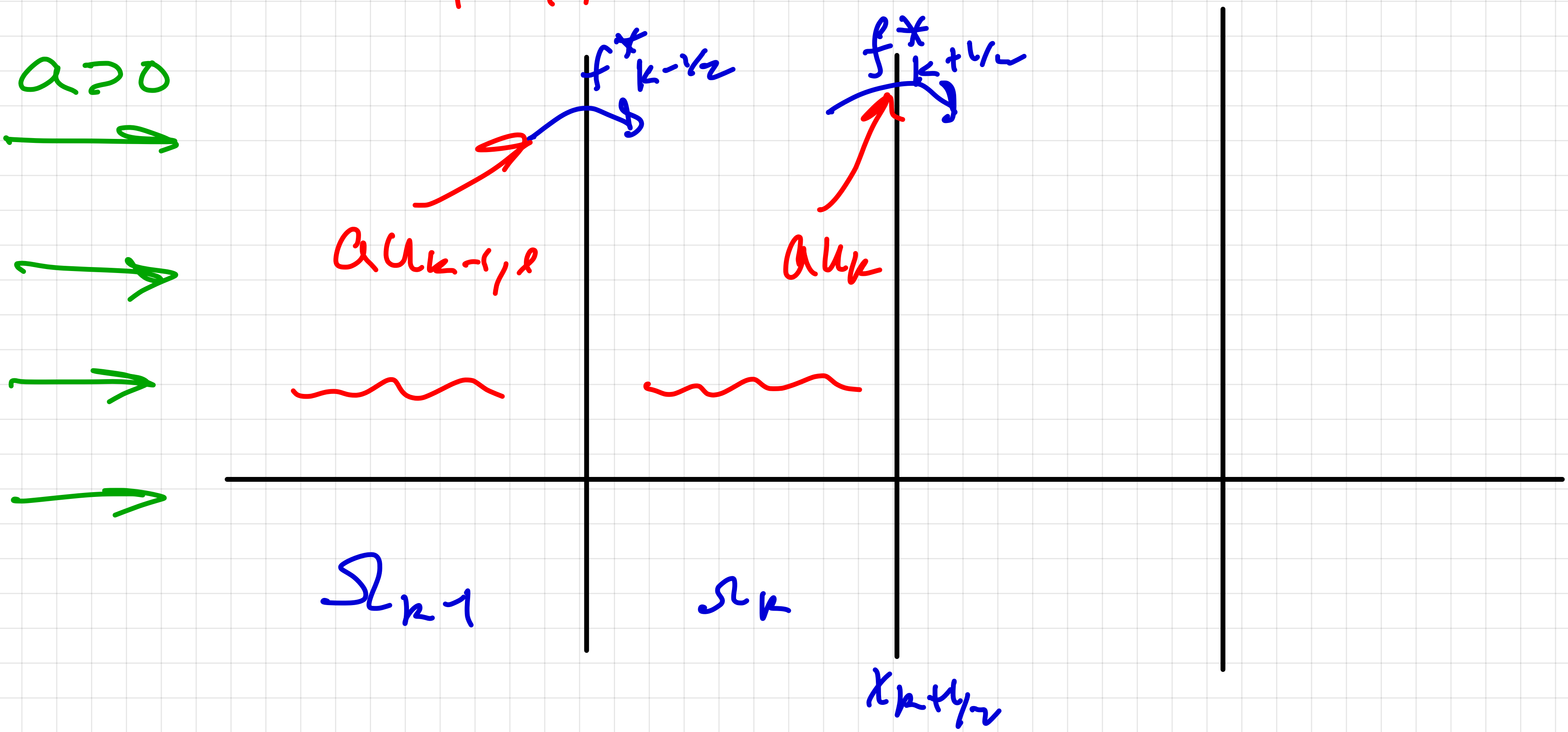
$f^*(u_{k,\ell}, u_{k+1,\ell}) = a u_{k,\ell}$

$f^*(u_{k-1,\ell}, u_{k,\ell}) = a u_{k-1,\ell}$

$a>0$ $\quad \dfrac{u_{k,l+1}-u_{k,l}}{h_t} + \dfrac{a u_{k,l} - a u_{k-1,l}}{h_x} = 0$

$f^*(u_{k,l}, u_{k+1,l}) = a u_{k,l}$

$f^*(u_{k-1,l}, u_{k,l}) = a u_{k-1,l}$



<u>if $a<0$</u> ETFS

$$\frac{u_{k,\ell+1} - u_{k\ell}}{h_t} + \frac{a u_{k+1,\ell} - a u_{k,\ell}}{h_x} = 0$$

$$f^*_{k+1/2,\ell} = a u_{k+1,\ell}$$

$=$ "upwind"

Q2 Write $f^*(u_{k,\ell}, u_{k+1,\ell})$

so that $a>0$ or $a<0$:

$a>0$: $f^* = a u_k$

$a<0$: $f^* = a u_{k+1}$

$$\rightarrow \quad f^*(u_{k,\ell}, u_{k+1,\ell}) = \frac{a u_{k,\ell} + a u_{k+1,\ell}}{2} - \frac{|a|}{2}\left(u_{k+1,\ell} - u_{k,\ell}\right)$$