Next: two remaining "issues"

① systems of conservation laws

② 2D, 3D ?

Today: 2/22

Systems.

General: $\underline{u} \in \mathbb{R}^p$ ← p terms

$$\frac{\partial \underline{u}}{\partial t} + \nabla \cdot F(\underline{u}) = 0$$

$$F(\underline{u}) = \begin{bmatrix} | & | & | \\ f(u) & g(u) & h(u) \\ | & | & | \end{bmatrix}$$

(3D)

$p \times 3$

div over "rows"

1st look at 1D: $\underline{u} \in \mathbb{R}^p$

$$\frac{\partial \underline{u}}{\partial t} + \frac{\partial \underline{f}(\underline{u})}{\partial x} = 0$$ "conservation form"

$$\Rightarrow \frac{\partial \underline{u}}{\partial t} + A(\underline{u}) \frac{\partial \underline{u}}{\partial x} = 0$$ "quasilinear form"

$$A(\underline{u}) = \frac{\partial f(\underline{u})}{\partial \underline{u}} \in \mathbb{R}^{p \times p}$$

= flux Jacobian

Definition 9.1: Hyperbolic system of conservation laws

The PDE system in the conservation law form of Equation (9.2) is called hyperbolic if the flux Jacobian $A(\mathbf{u}) \in \mathbb{R}^{p \times p}$ has p real eigenvalues and a full set of p linearly independent eigenvectors.

→ p eigenvalues ~ p waves

~ p characteristics

Simple case

let $\underline{f}(\underline{u}) = A\,\underline{u}$

$A \in \mathbb{R}^{p\times p}$

→ Assume hyperbolic

→ p linearly independent eigenvectors

p eigenvalues

→ diagonalizable

let $AR = R\Lambda$

Λ eigenvalues

R eigenvectors

$$AR = R\Lambda$$

$$R = \begin{bmatrix} | & | & & | \\ r_1 & r_2 & \cdots & r_p \\ | & | & & | \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_p \end{bmatrix}$$

$$A = R\Lambda \underset{\substack{\uparrow \\ L}}{R^{-1}}$$

$$= R\Lambda L$$

original problem:

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = 0$$

$$\Downarrow$$

$$L \left(\frac{\partial u}{\partial t} + R \Lambda L \frac{\partial u}{\partial x} = 0 \right)$$

$$\Downarrow$$

$$\frac{L \partial u}{\partial t} + \Lambda L \frac{\partial u}{\partial x} = 0$$

$\underline{w} = L \underline{u}$

$$\Downarrow$$

$$\frac{\partial w}{\partial t} + \Lambda \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial w}{\partial t} + \underset{\uparrow}{\Lambda} \frac{\partial w}{\partial x} = 0$$

diagonal

$\Rightarrow$ p independent equations:

$$\frac{\partial w_i(x,t)}{\partial t} + \lambda_i \frac{\partial w_i(x,t)}{\partial x} = 0$$

$\Rightarrow$ solution?

$$w_i(x,t) = \underset{\uparrow}{w_i^{(0)}}(x - \lambda_i t)$$

initial data

$$\boxed{\underset{\sim}{w} \text{ are called the } \underline{\text{characteristic}} \text{ variables}}$$

Also: $\underline{w} = R^{-1} \underline{u}$

$\rightarrow \quad \underline{u} = \underline{R}\,\underline{w}$

we know these!

$$= \begin{bmatrix} | & | & & | \\ r_1 & r_2 & \cdots & r_p \\ | & | & & | \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_p \end{bmatrix}$$

$$= \sum_{i=1}^{p} w_i r_i$$

$$= \sum_{i=1}^{p} w_i^{(0)} (x - \lambda_i t)\, r_i$$

Example

$$\begin{cases} u_t + u_x + 3v_x = 0 \\ v_t + 3u_x + v_x = 0 \end{cases}$$

with $\underline{u}(x,0) = \begin{cases} \underline{u}^- & x < 0 \\ \underline{u}^+ & x \geq 0 \end{cases}$

$$\underline{u}^- = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$u^+ = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Steps

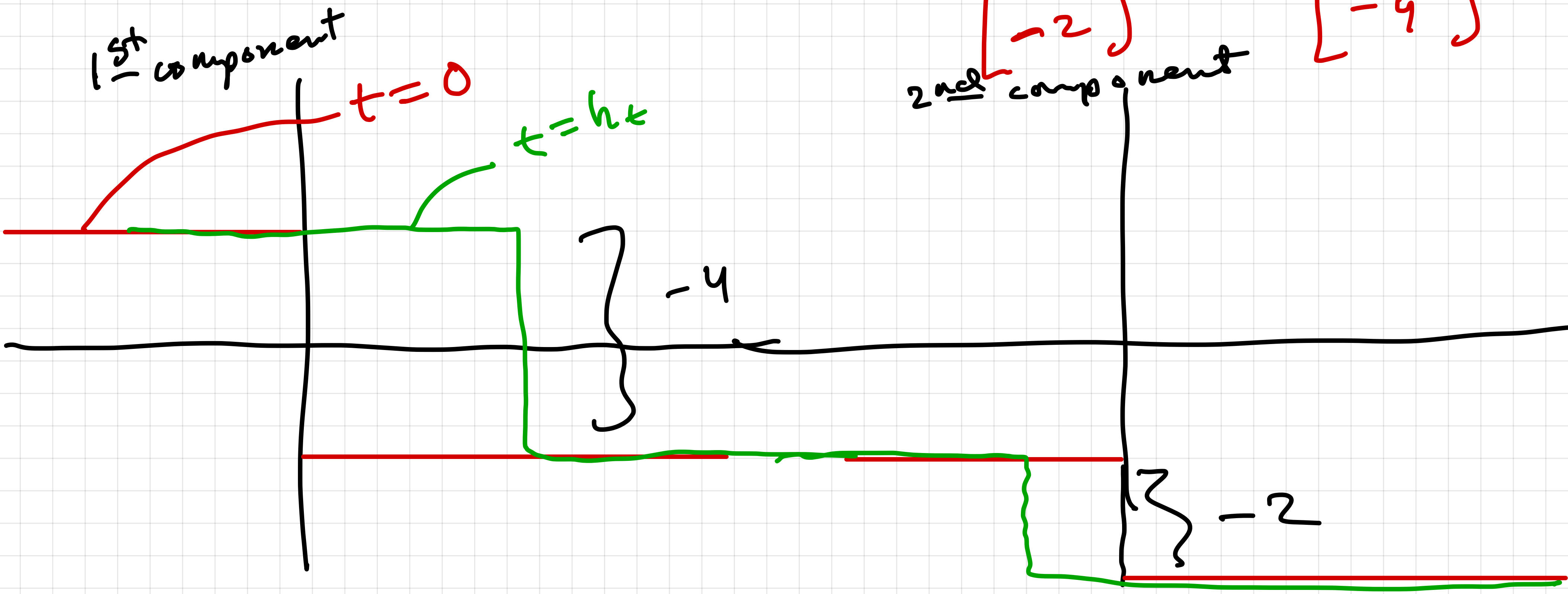
1. Write the problem in terms of $\underline{w}$
2. Draw the solutions in $\underline{w}$
3. Write the solution in $\underline{u}$
4. Draw the solution in $\underline{u}$

$$u_t + \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} u_x = 0 \qquad \lambda = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

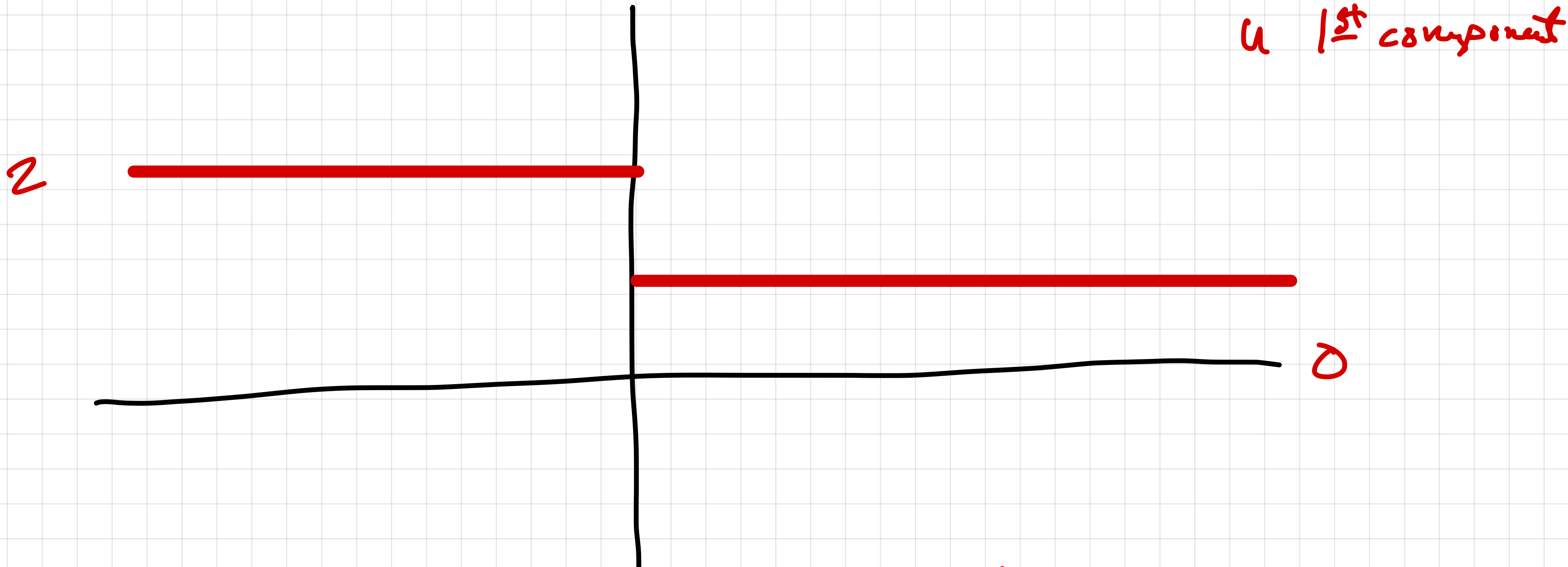
$$u^- = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad u^+ = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$A \to R = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$w = L\underline{u} \to \quad w^- = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \quad w^+ = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$



let $\underline{u} = R\,\underline{w}$



look at the "jump" in u:

$$\Delta u = u^+ - u^- = \begin{bmatrix}1\\-3\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}-1\\-3\end{bmatrix}$$

$$\to \Delta w = L\Delta u = \begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}-1\\-3\end{bmatrix} = \begin{bmatrix}-4\\-2\end{bmatrix}$$

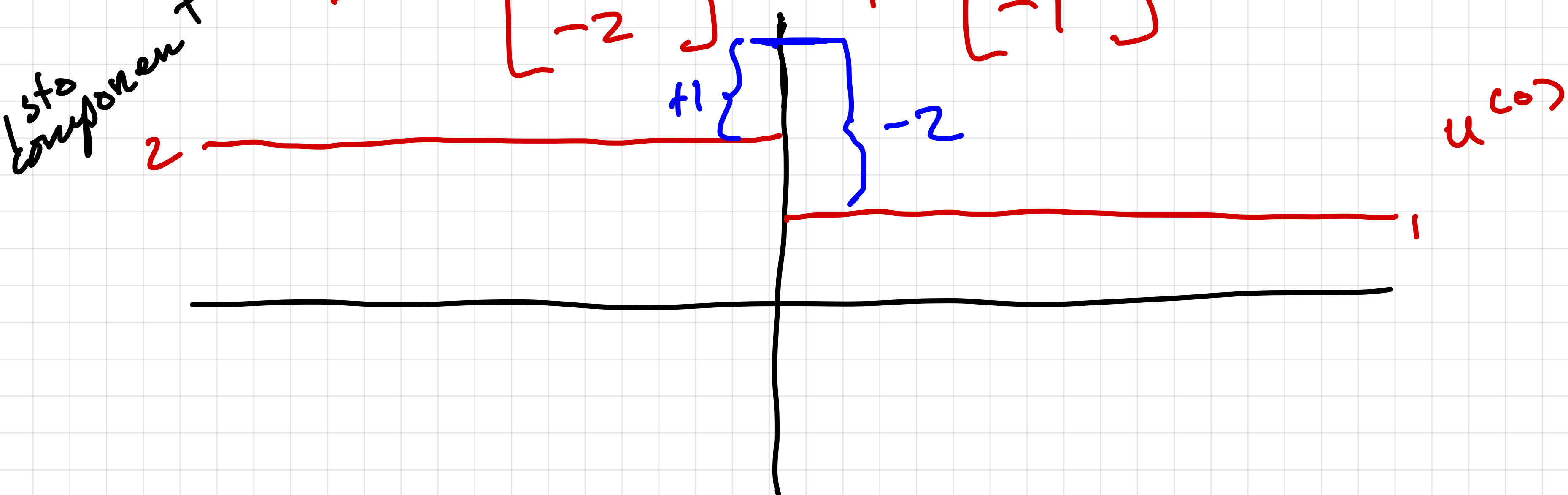
$$\Delta u = R \, \Delta w$$

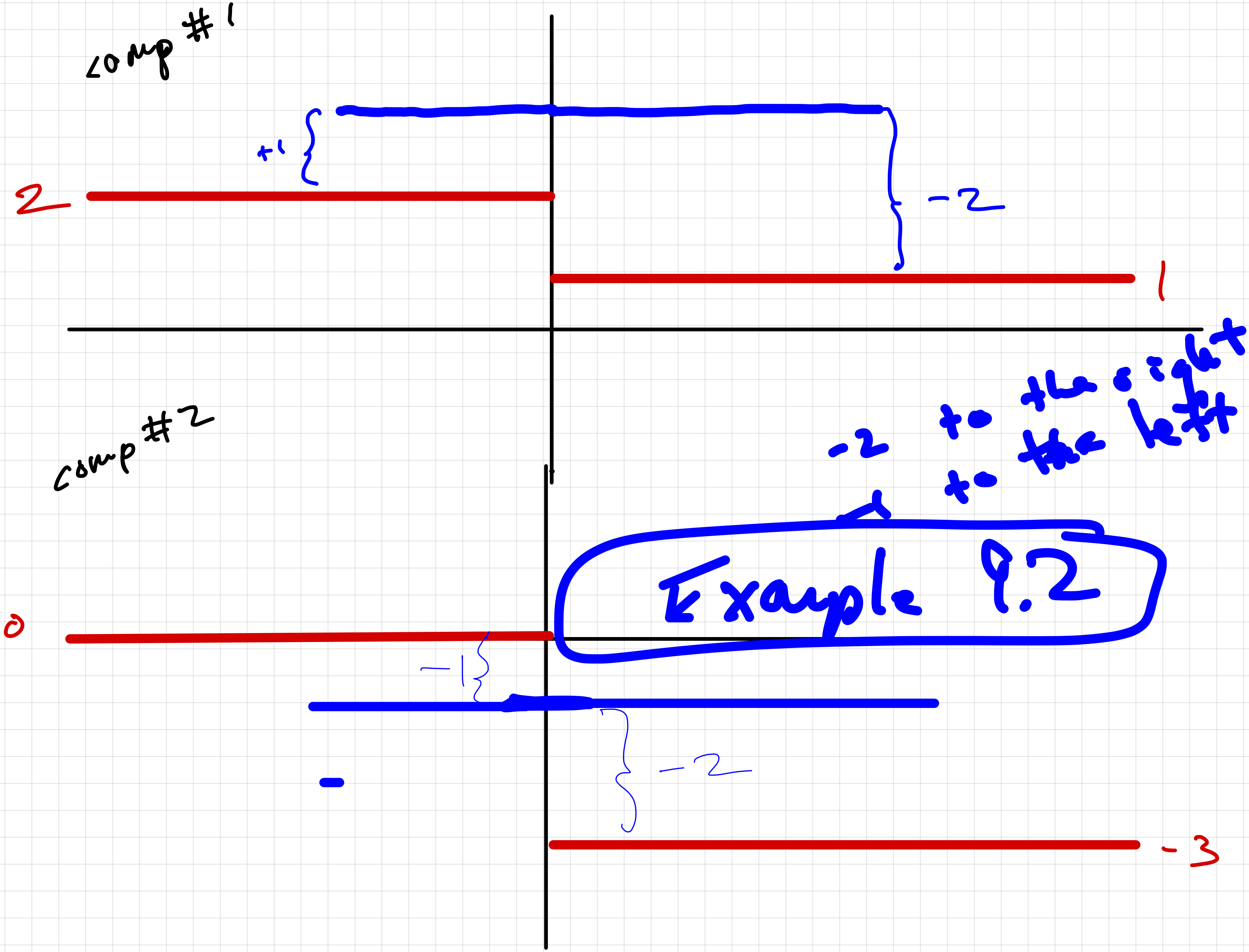
$$= \sum_{i=1}^{p=2} \Delta w_i \, r_i$$

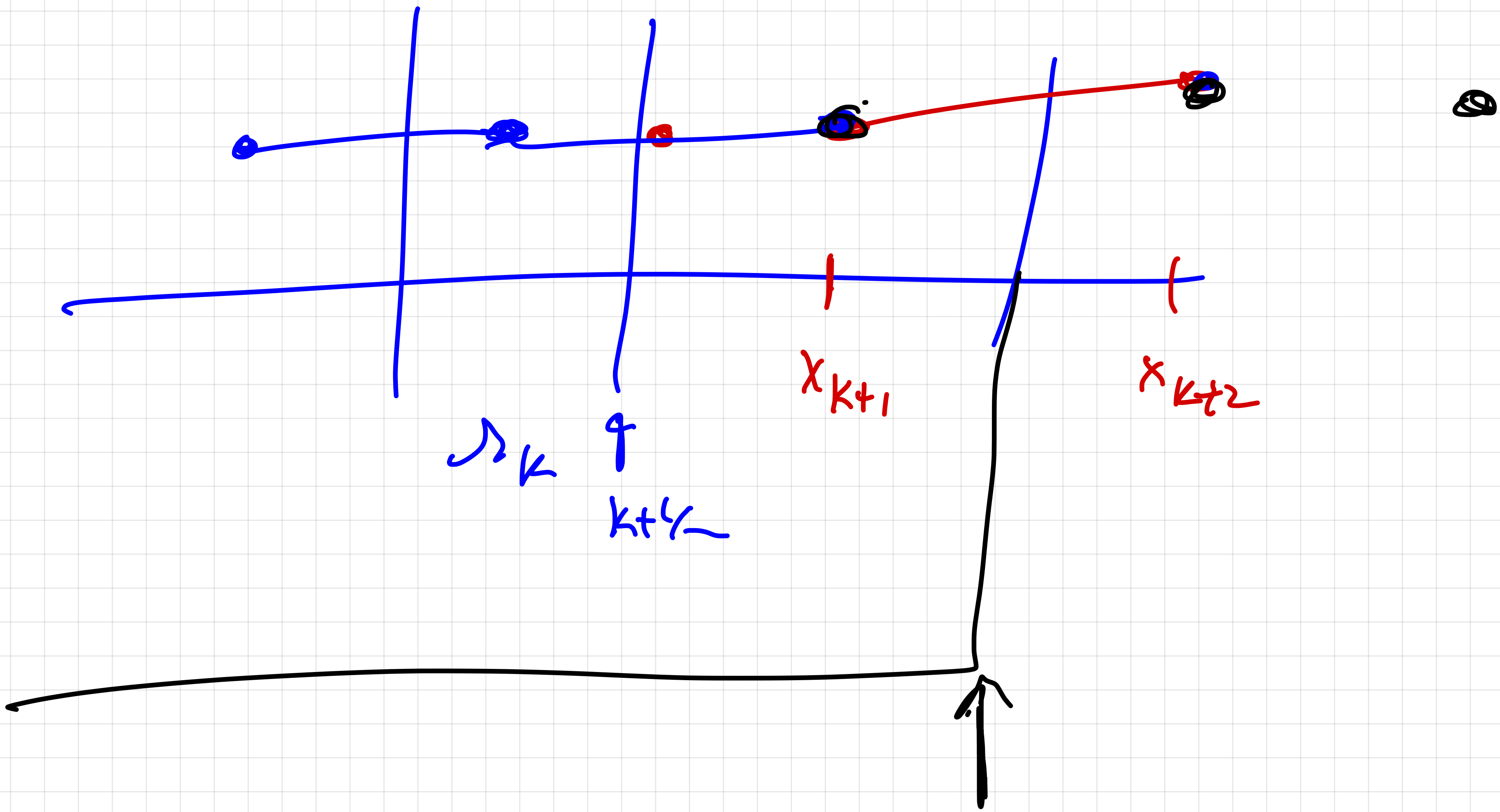
$$= \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (-4) + \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} (-2)$$

$$= \sum_{i=1}^{2} \Delta u_i$$

$$= \begin{bmatrix} -2 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$




comp #1
+1
2
-2
1
comp #2
-2
to the right
to the left
Example 9.2
0
-1
-2
-3



x_{k+1}
x_{k+2}