Last time: $\underline{u}_t + (F(\underline{u}))_x = 0$

$\rightarrow \underline{u}_t + A(u)\, u_x = 0$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\uparrow$

flux Jacobian

$A(u) = \dfrac{\partial f}{\partial \underline{u}} \in \mathbb{R}^{p \times p}$

Take $A \Rightarrow AR = R\Lambda$

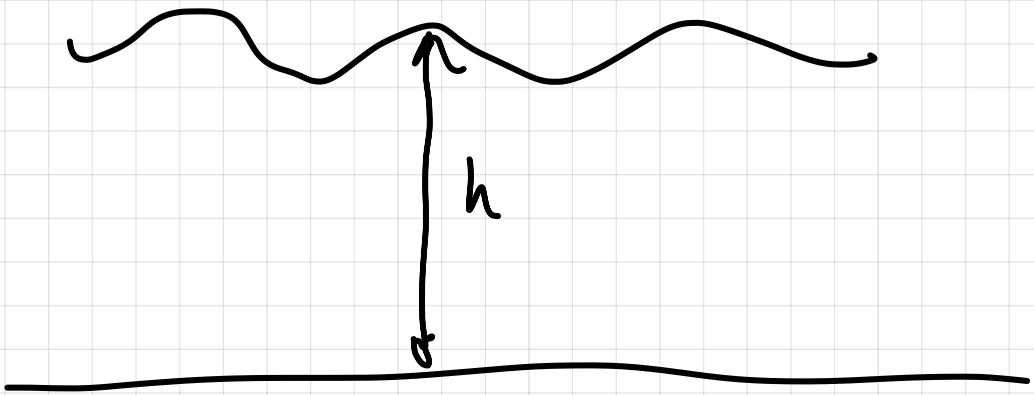
Shallow Water Equations

conservation mass + momentum

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ hu \\ hv \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} hu \\ hu^2+\frac{1}{2}gh^2 \\ huv \end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} hv \\ huv \\ hv^2+\frac{1}{2}gh^2 \end{bmatrix} = 0$$

h = water height

h, u, v : primitive variables



let $\underline{u} = \begin{bmatrix} h \\ m_x \\ m_y \end{bmatrix}$: the conserved variables

$m_x = h \cdot u$

$m_y = h \cdot v$

1D: (zero y momentum)

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ m_x \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} m_x \\ \frac{m_x^2}{h} + \frac{1}{2}gh^2 \end{bmatrix} = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ -\left(\frac{m_x}{h}\right)^2 + gh & \frac{2m_x}{h} \end{bmatrix}$$

$\frac{\partial}{\partial h}$

non-zero momentum:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\left(\frac{m_x}{h}\right)^2 + gh & \frac{2m_x}{h} & 0 \\ -\frac{m_x m_y}{h^2} & \frac{m_y}{h} & \frac{m_x}{h} \end{bmatrix}$$

$\rightarrow$ eigs are $\frac{m_x}{h} \pm \sqrt{gh}$, $\frac{m_x}{h}$

LLF

Fou: $f^*(u_{k\ell}, u_{k+1\ell}) = \dfrac{a u_{k\ell} + a u_{k+1\ell}}{h_x} - \dfrac{|a|}{2}\left(u_{k+1\ell} - u_{k\ell}\right)$

LF: $f^*(u_{k\ell}, u_{k+1\ell}) = \dfrac{f(u_{k\ell}) + f(u_{k+1\ell})}{h_x} - \dfrac{\alpha_{k+1/2}}{2}\left(u_{k+1\ell} - u_{k\ell}\right)$

$\alpha_{k+1/2} = \max\left(|f'(u_{k\ell})|, |f'(u_{k+1,\ell})| \right)$

For systems!

$$\underset{\sim}{f}^{*}(\underset{\sim}{u}_{k\ell}, \underset{\sim}{u}_{k+1\ell}) = \frac{\underset{\sim}{f}(\underset{\sim}{u}_{k\ell}) + \underset{\sim}{f}(\underset{\sim}{u}_{k+1\ell})}{h_x} - \frac{\alpha_{k+1/2}}{2}(\underset{\sim}{u}_{k+1\ell} - \underset{\sim}{u}_{k\ell})$$

$$\alpha_{k+1/2} = \max\left(\max_j |\lambda_j(u_{k\ell})|, \max_j |\lambda_j(u_{k+1,\ell})| \right)$$

2D ?

$$u = u(x, y, t)$$

scalar: $\quad u_t + \nabla \cdot \underline{F}(u) = 0$

$$\underline{F} = \begin{bmatrix} f(u) \\ g(u) \end{bmatrix}$$

$$\rightarrow \quad u_t + (f(u))_x + (g(u))_y = 0$$

$$\rightarrow \quad \frac{u_{jk\ell+1} - u_{jk\ell}}{h_t} + \frac{f^*(u_{jk\ell}, u_{j+1k\ell}) - f^*(u_{j-1k\ell}, u_{jk\ell})}{h_x}$$

$$+ \frac{g^*(u_{jk\ell}, u_{jk+1\ell}) - g^*(u_{jk-1\ell}, u_{jk\ell})}{h_y}$$

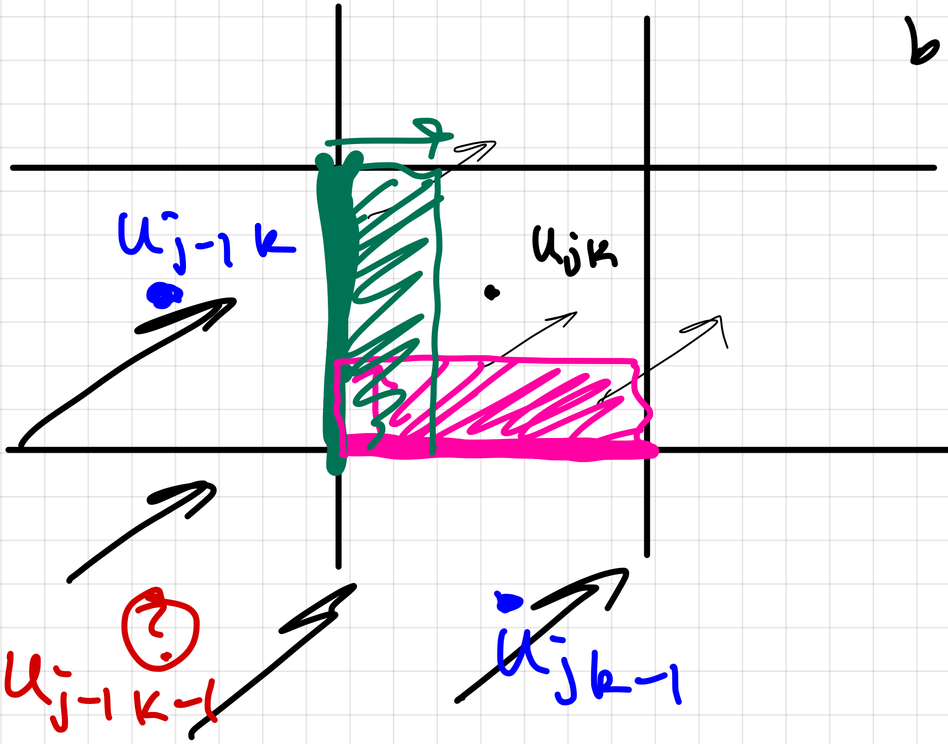
$$= 0$$

Option #1 Solve Riemann problems:

assume $(f(u))_x + (g(u))_y = [a,b]\cdot\nabla u$

$a>0$

$b>0$



one problem: $\uparrow b$

one problem: $\longrightarrow a$

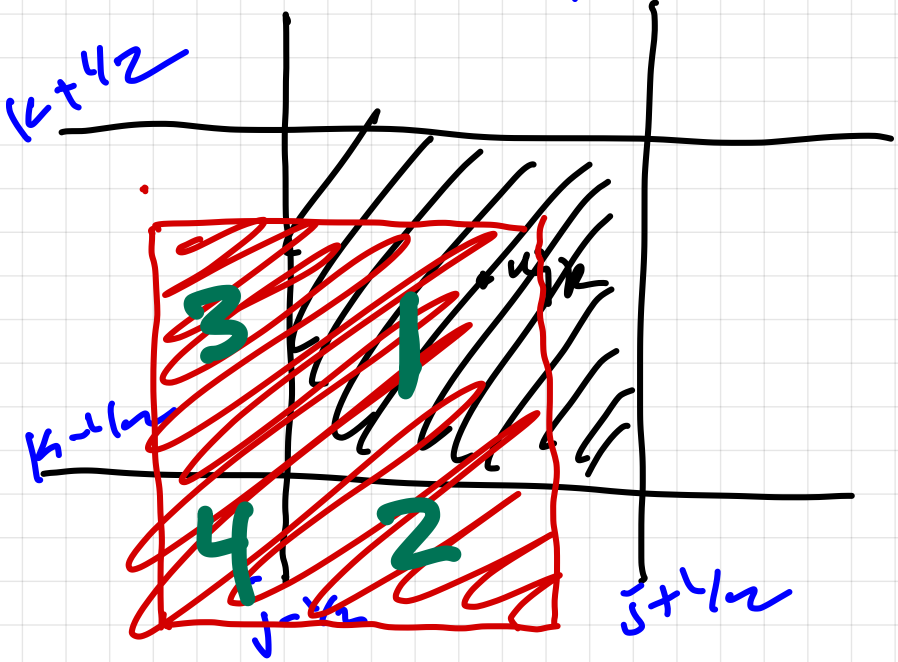
u_{jk} depends on u_{jk}, u_{j-1k}, u_{jk-1}

$\rightarrow$ $$u_{jk\,\ell+1} = u_{jk\ell} - \frac{a h_t}{h_x}\left(u_{jk} - u_{j-1\,k}\right) - \frac{b h_t}{h_y}\left(u_{jk} - u_{jk-1}\right)$$

$\rightarrow$ stable if $\left|\frac{a\,\Delta t}{\Delta x}\right| + \left|\frac{b\,\Delta t}{\Delta x}\right| < 1$

Option #2 REA

let $\tilde{u}$ = piecewise constant



$$u_{jk}^{l+1} = \frac{1}{h_x h_y} \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{y_{k-1/2}}^{y_{k+1/2}} \tilde{u}(x - a h_t, y - b h_t, t_l)\, dx\, dy$$

$$= \frac{1}{h_x h_y} \int_{x_{j-1/2} - a h_t}^{x_{j+1/2} - a h_t} \int_{y_{k-1/2} - b h_t}^{y_{k+1/2} - b h_t} \tilde{u}(x, y, t_l)$$

$$= \frac{1}{h_x h_y} \Big[(h_x - a h_t)(h_y - b h_t)\, u_{jk}^l \quad ①$$

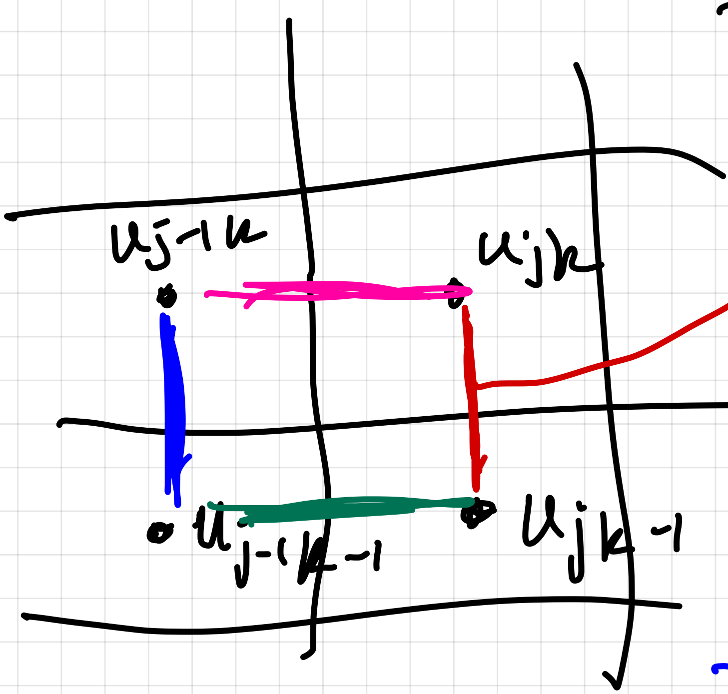
$$+ (h_x - a h_t)\, b h_t\, u_{j k-1}^l \quad ②$$

$$+ a h_t \cdot (h_y - b h_t)\, u_{j-1 k}^l \quad ③$$

$$+ a h_t\, b h_t\, u_{j-1 k-1}^l \quad ④$$

let $\lambda_x = \frac{a h_t}{h_x}$ $\quad \lambda_y = \frac{b h_t}{h_x}$

$$
\begin{aligned}
u_{jk\ell+1} = u_{jk\ell} &- \lambda_x \left(u_{jk\ell} - u_{j-1k\ell} \right) \\
&- \lambda_y \left(u_{jk\ell} - u_{jk-1\ell} \right) \\
&- \frac{\lambda_x \lambda_y}{2} \Big[\left(u_{jk\ell} - u_{jk-1\ell} \right) \\
&\qquad - \left(u_{j-1k\ell} - u_{j-1\,k-1\ell} \right) \\
&\qquad + \left(u_{jk\ell} - u_{j-1k\ell} \right) \\
&\qquad - \left(u_{jk-1\ell} - u_{j-1k-1\ell} \right) \Big]
\end{aligned}
$$



$\rightarrow$ stable if $\max\left(\lambda_x, \lambda_y\right) < 1$

$u = \text{np linspace}(0, 1, \text{ncells}+2)$