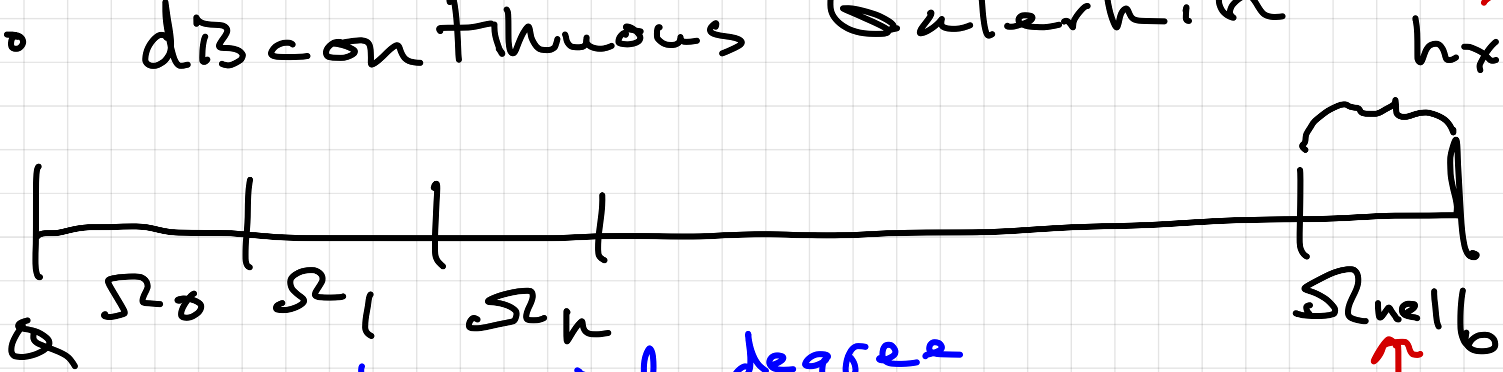
Today 3/1

- discontinuous Galerkin

1D mesh



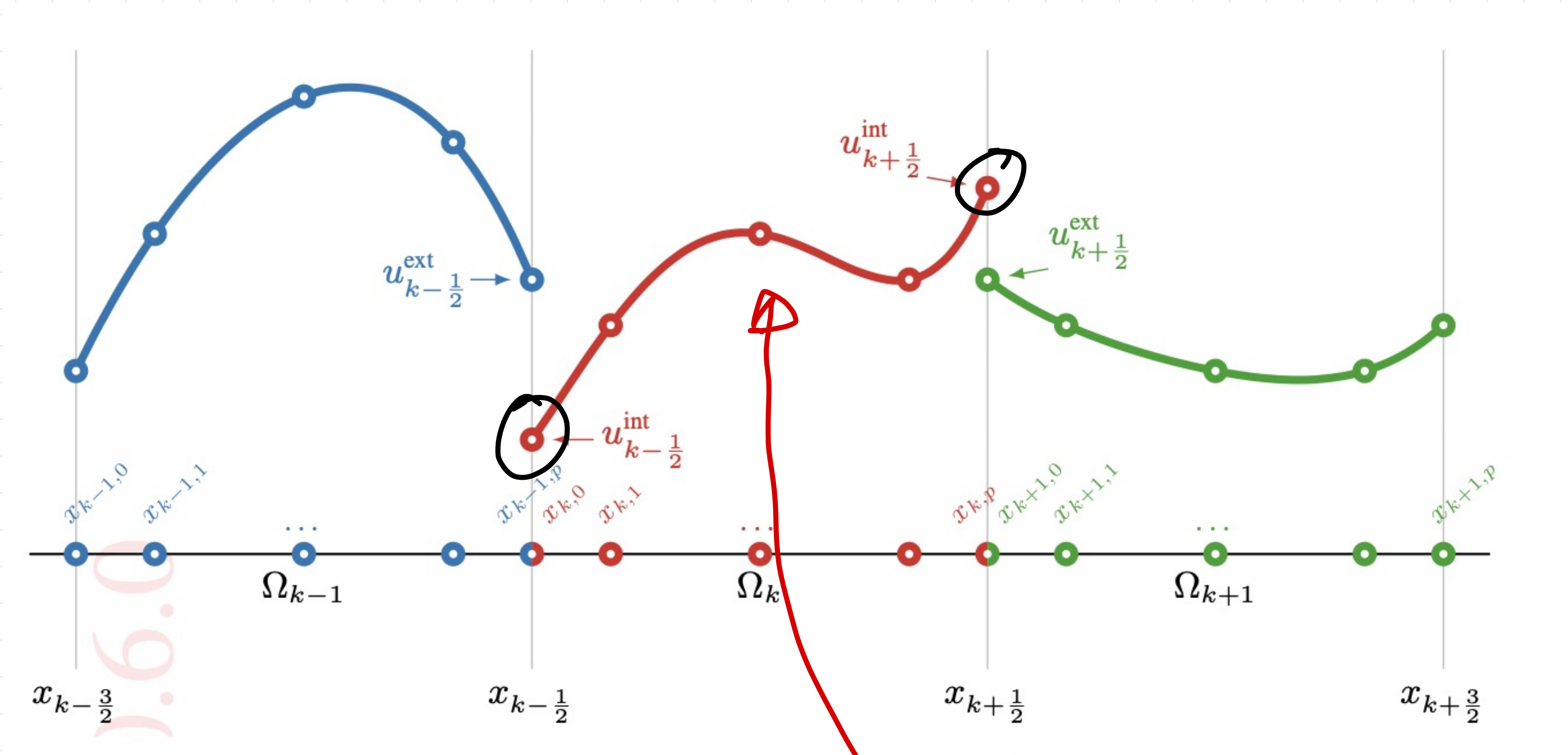
elements

polynomial degree

$$V_{h_x}^{p} = \left\{ v(x) \;\middle|\; v\big|_{\Omega_k} \in \mathcal{P}^{p}(\Omega_k) \quad \forall k \right\}$$

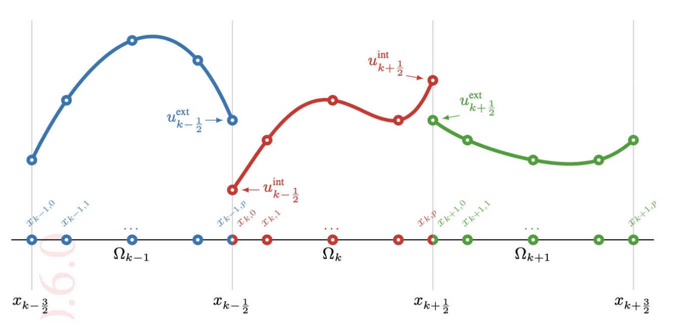
mesh size

space of p-deg. polynomials on Ω_k



$p = 4$

How to represent each $v(x)\big|_{\Omega_k}$?



o <u>modal representation</u>

e.g. $v(x)\big|_{\Omega_k} = \sum_{q=0}^{p} c_q \cdot x^q$

or

$v(x)\big|_{\Omega_k} = \sum_{q=0}^{p} c_q \cdot L_q(x)$

Legendre

easily take integrals and derivatives

o <u>nodal representation</u>

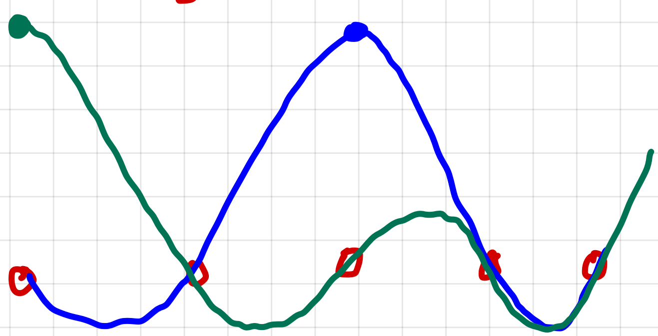
↳ ① set of nodes

② Lagrange interpolant over these nodes.

$f(x) \to \sum_{q=0}^{p} f(x_i) \cdot \ell_q(x)$

Lagrange

Lagrange : p :

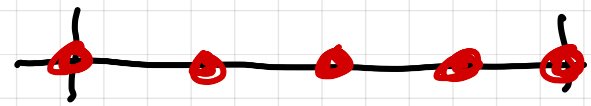


$p=4$

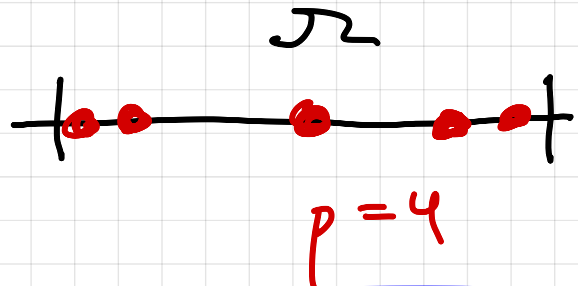
easily evaluate

Picking a set of nodes

- evenly spaced



- Gauss nodes



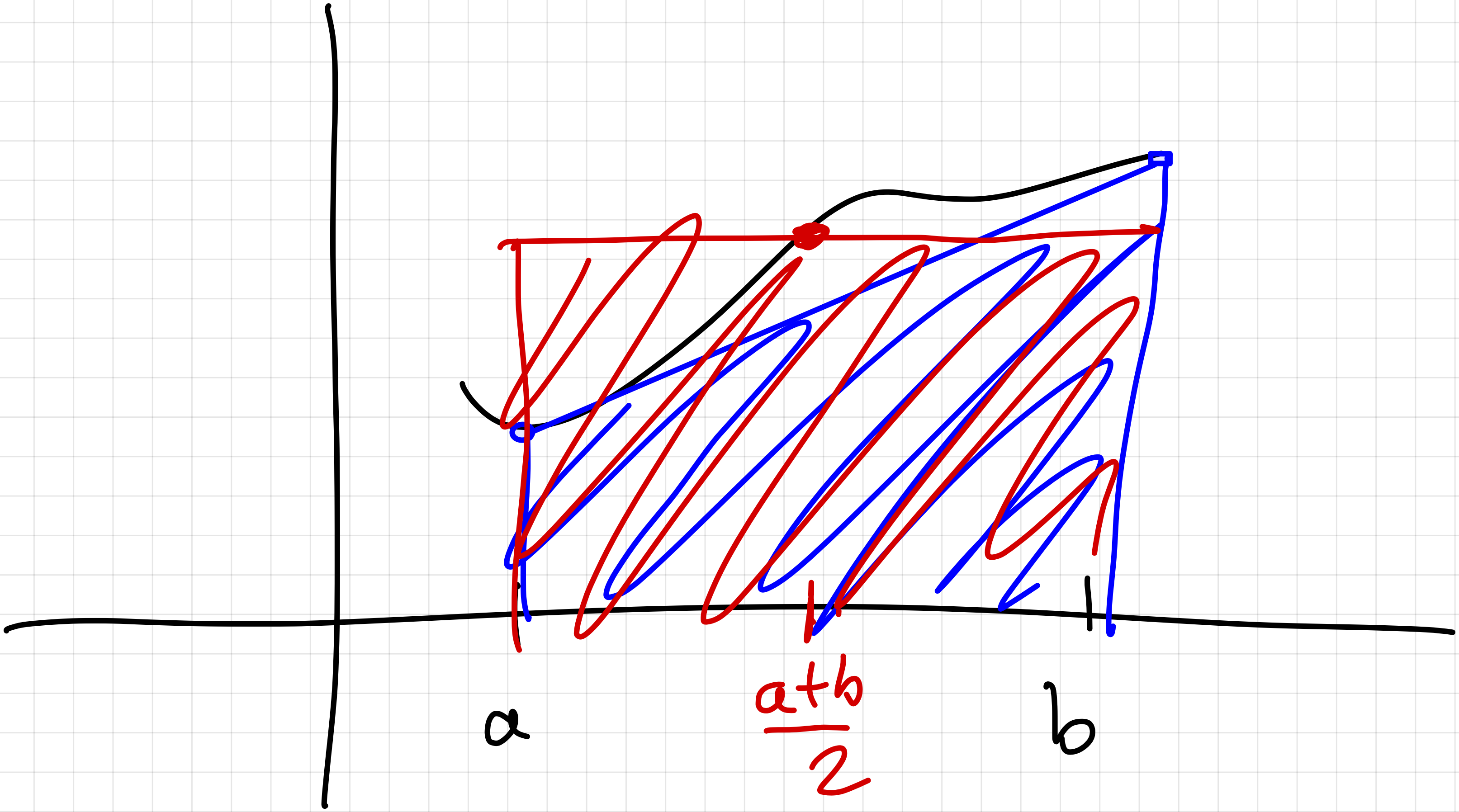
Quadrature

want $\displaystyle\int_{-1}^{1} f(x)\,dx$

approximate by $\displaystyle\sum_{i=0}^{m} w_i f(x_i)$

w_i: weights

x_i: quadrature nodes


a
a+b
2
b

Gauss Quadrature

3-pt scheme: $\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$

$\rightarrow$ 6 degrees of freedom

w_1, w_2, w_3

x_1, x_2, x_3

make this scheme exact

for

$$\left.\begin{aligned} f(x) &= 1 \\ f(x) &= x \\ f(x) &= x^2 \\ f(x) &= x^3 \\ f(x) &= x^4 \\ f(x) &= x^5 \end{aligned}\right\} \text{6 constraints}$$

We will use GLL nodes:

Gauss-Legendre-Lobatto node

± 1, $p-1$ roots of $\dfrac{d L_p(x)}{dx}$

Legendre

Write the weak form of $u_t + (f(u))_x = 0$

on Ω_k:

linear algebra. want to solve

$$A\underline{x} = \underline{b}$$

① Find x s.t.

$$\underline{b} - A\underline{x} = 0$$

② F x s.t.

$$\underline{v}^T\underline{b} - \underline{v}^T A\underline{x} = 0 \quad \forall \underline{v}$$

$$\downarrow$$

$$\underline{v}^T(\underline{b} - A\underline{x}) = 0$$

$$\int_{\Omega_k} \left(u_t + (f(u))_x\right) v \, dx = 0 \quad \forall v \in \cancel{V^p_{h_x}} \; P^p(\Omega_k)$$

$$\int_{\Omega_k} \left(u_t + (f(u))_x \right) v \, dx$$

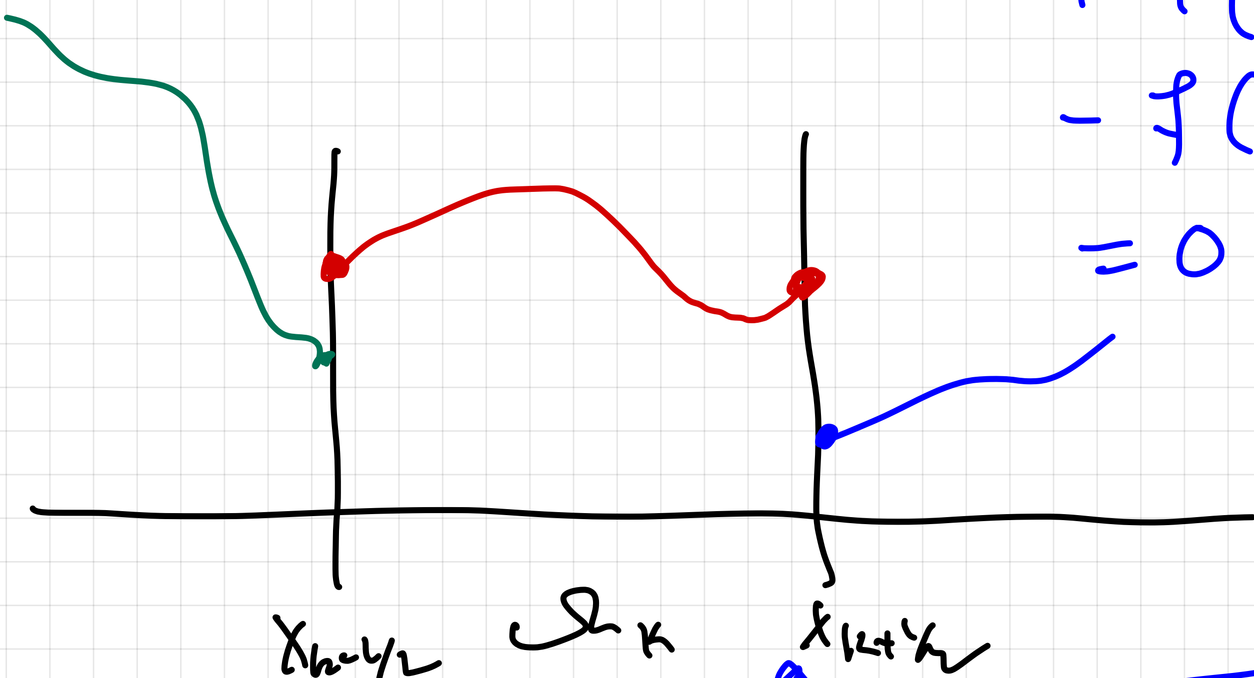
I.B.P. on "x":

$$\int_{\Omega_k} u_t \, v \, dx - \int_{\Omega_k} f(u) \, v_x \, dx$$

$$+ f\left(u(x_{k+1/2}, t)\right) v(x_{k+1/2})$$

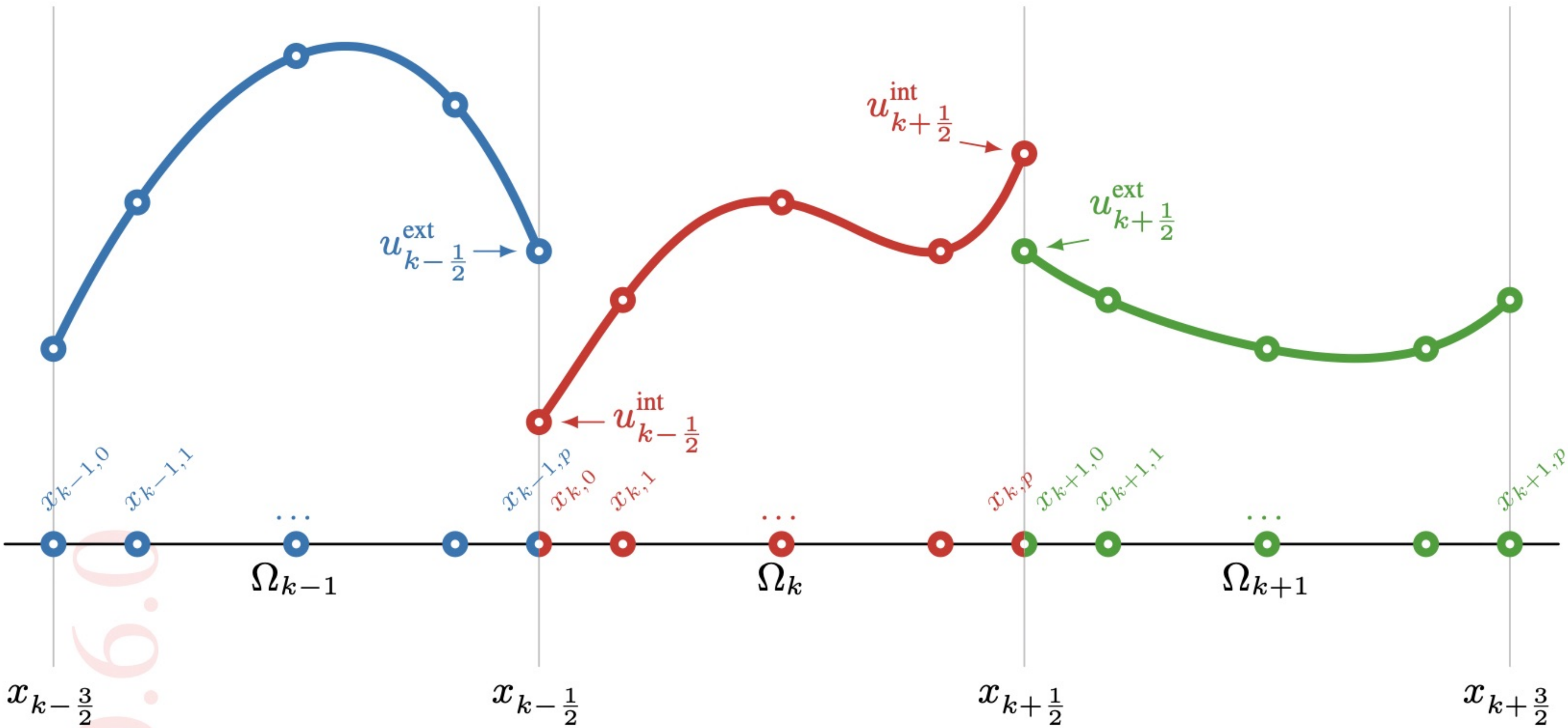
$$- f\left(u(x_{k-1/2}, t)\right) v(x_{k-1/2})$$

$$= 0$$



what about

$u(x_{k-1/2}, t)$?



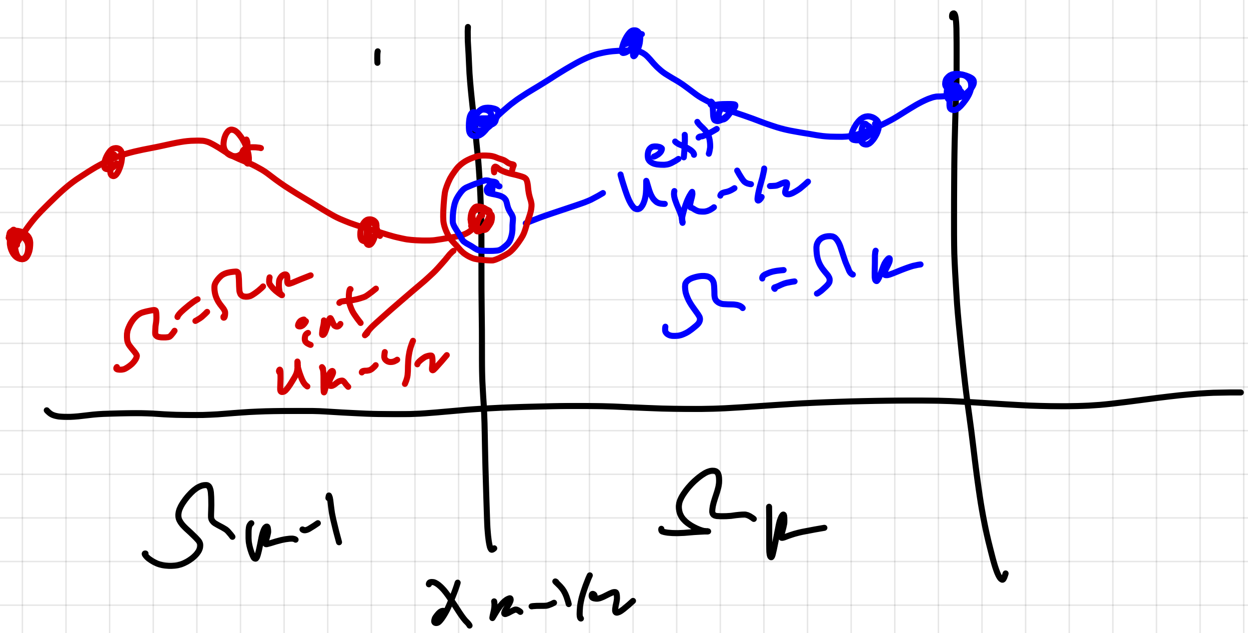
Ω_k

$f^*(u^{int}_{k-1/2}, u^{ext}_{k-1/2})$ represents the value of f at $x_{k-1/2}$

$$\int_{\Omega_k} u_t v - f(u)\, v_x \, dx$$
$$+ f^*\left(u^{int}_{k+1/2}, u^{ext}_{k+1/2}, \Omega_k\right) v(x_{k+1/2})$$
$$- f^*\left(u^{int}_{k-1/2}, u^{ext}_{k-1/2}, \Omega_k\right) v(x_{k-1/2})$$
$$= 0$$

let $a > 0$

FOU: $f^*\left(u^{int}_{k-1/2}, u^{ext}_{k-1/2}, \Omega\right) = \begin{cases} a\, u^{ext}_{k-1/2}, & \Omega = \Omega_k \\ a\, u^{int}_{k-1/2}, & \Omega = \Omega_{k-1} \end{cases}$



introduce two notational things:

average: $$\{a\, u_{k-1/2}\} = \frac{a\, u^{int}_{k-1/2} + a\, u^{ext}_{k-1/2}}{2}$$

jump: $$[u_{k-1/2}] = n^{int}_{k-1/2}\, u^{int}_{k-1/2} + n^{ext}_{k-1/2}\, u^{ext}_{k-1/2}$$

outward normal

"upwind":

$$f^*(x^{int}_{k-1/2}, x^{ext}_{k-1/2}) = \{a\, u_{k-1/2}\} + \frac{a}{2}[u_{k-1/2}]_{\Omega}$$

cell centered