Today 3/6

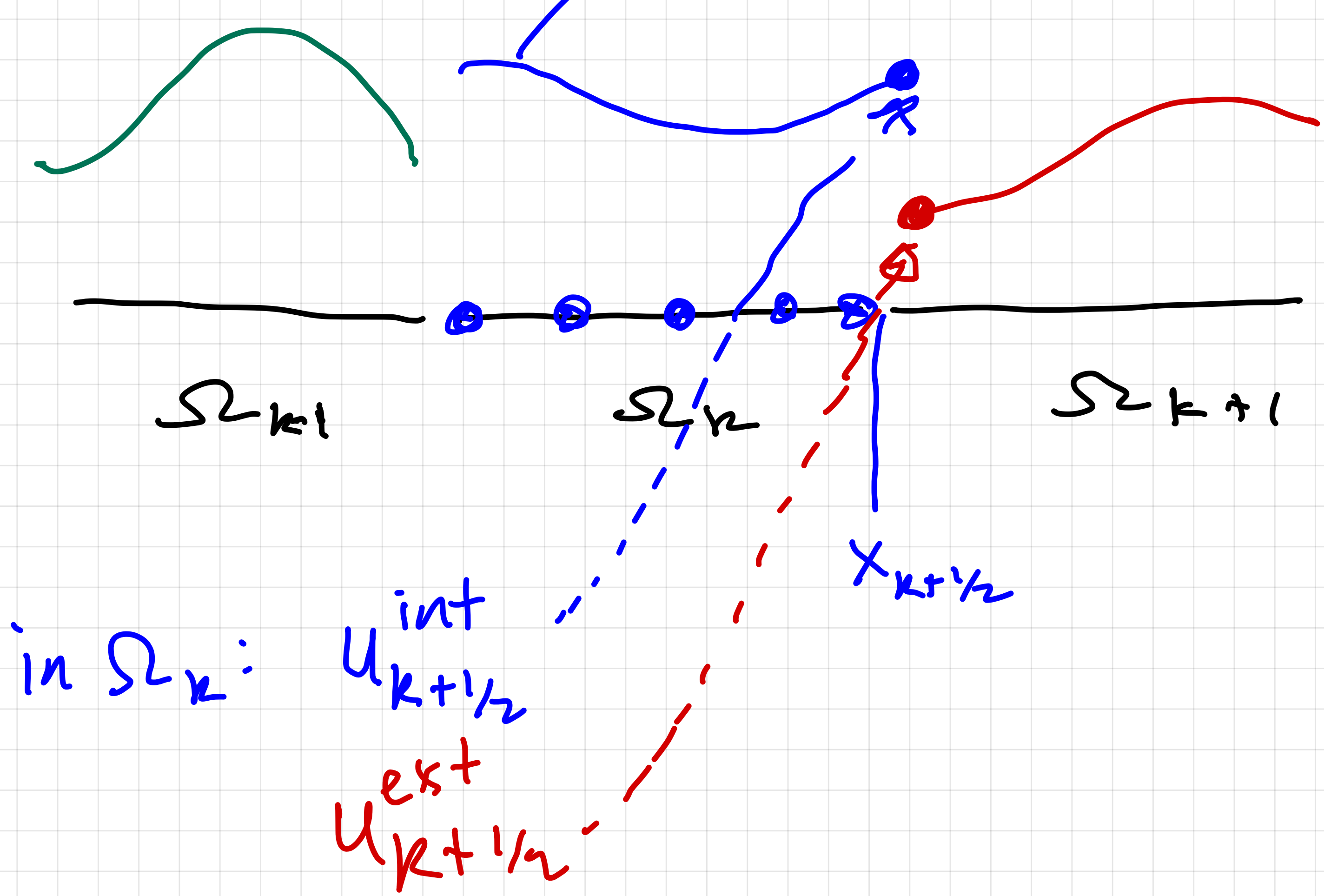
- DG, implementation

Wed

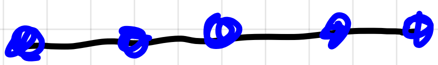
- Finite Elements
- Project overview

Last time

- move to an element centric view



$p = 4$ $\Rightarrow$ 5 d.o.f.s



Ω_k

∘ represent the "local" polynomial (polynomial on element Ω_k) as either ① modal representation

if $u(x) \in P^4(\Omega_k)$

then $u(x) = \sum_{i=0}^{4} \tilde{u}_i \, \psi_i(x)$ coefficients

$\psi_i(x) =$ Legendre

② nodal representation

$$\phi_i(x) = \frac{\prod_{i \neq j, 0}^{p} (x - x_j)}{\prod_{i \neq j, 0} (x_i - x_j)}$$

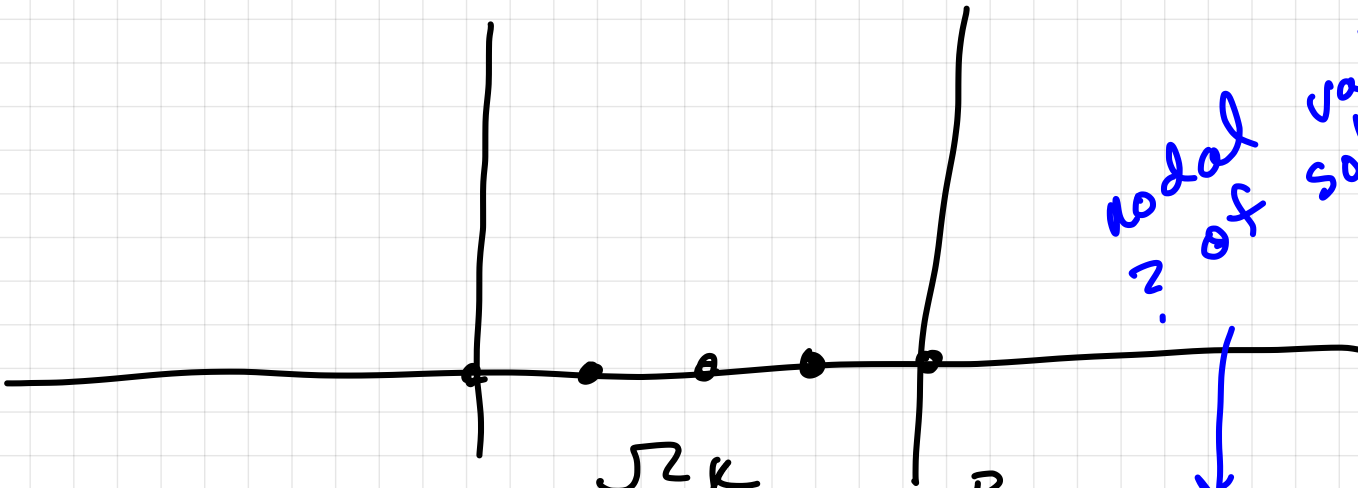
if $u \in P^4(\Omega_k)$

then $u(x) = \sum_{i=0}^{4} u(x_i) \cdot \phi_i(x)$

nodal values

$\phi_i(x) =$ Lagrange.

$u_t + (f(u))_x = 0$



nodal values of solution

degree p Lagrange

let $u(x,t)\big|_{\Omega_k} = \sum_{q=0}^{p} u_{k,q}(t) \cdot \phi_{k,q}(x)$

Goal: find $u_{k,q}$ for $k = 1, \dots, \#\text{elements}$

$q = 0, \dots, p$

let $f(u(x,t))\big|_{\Omega_k} = \sum_{q=0}^{p} f_{k,q}(t)\, \phi_{k,q}(x)$

general: $f_{k,q}(t) = f(u_{k,q}(t))$

Back to the weak form of

$$u_t + (f(u))_x = 0$$

Find u s.t.

$$\int_{\Omega_k} u_t v - \int_{\Omega_k} f(u)\, v_x + f(u(x_{k+1/2}))v - f(u(x_{k-1/2}))v = 0 \quad \forall v.$$

→ replace with

$$u(x,t) = \sum_{q=0}^{p} u_{k,q} \phi_{kq}$$

$$f(u(x,t)) = \sum_{q=0}^{p} f_{k,q} \phi_{kq}$$

Find $u_{k,q}$ (coefficients) such that

$$\int_{\Omega_k} \sum_{q=0}^{p} \frac{d u_{kq}(t)}{dt} \phi_{kq} \phi_{kr} \, dx - \int_{\Omega_k} \sum_{q=0}^{p} f_{kq}(t) \cdot \phi_{kq} \frac{d\phi_{kr}}{dx} dx + f_{k+1/2} \phi_r - f_{k-1/2} \phi_r = 0 \quad \forall \phi_{kr}$$

at time t:

$$\int_{\Omega} \frac{du(x,t)}{dt}\, v(x)$$

Find $u \in P^{P}(\Omega_k)$

$$\int_{\Omega_k} \left(u_t + (f(u))_x\right) v\, dx = 0 \quad \forall\, v \in P^{P}(\Omega_k)$$

let $\{\phi_q\}$ be a basis for $P^{P}(\Omega_k)$

Find $u = \sum u_i \phi_i$ st.

$$\int_{\Omega_k} \left(u_t + (f(u))_x\right) \phi_j\, dx = 0 \quad \forall\, \phi_j \in \{\phi_q\}$$

linear algebra

A nonsingular

Find $\underline{x} \in \mathbb{R}^n$ s.t.

$$A\underline{x} = b \in \mathbb{R}^n$$

Find $\underline{x} \in \mathbb{R}^n$ st.

$$b - A\underline{x} = 0$$

Find $\underline{x} \in \mathbb{R}^n$ st.

$$\underline{v}^T (b - A\underline{x}) = 0 \quad \forall\ \underline{v} \neq 0 \in \mathbb{R}^n$$

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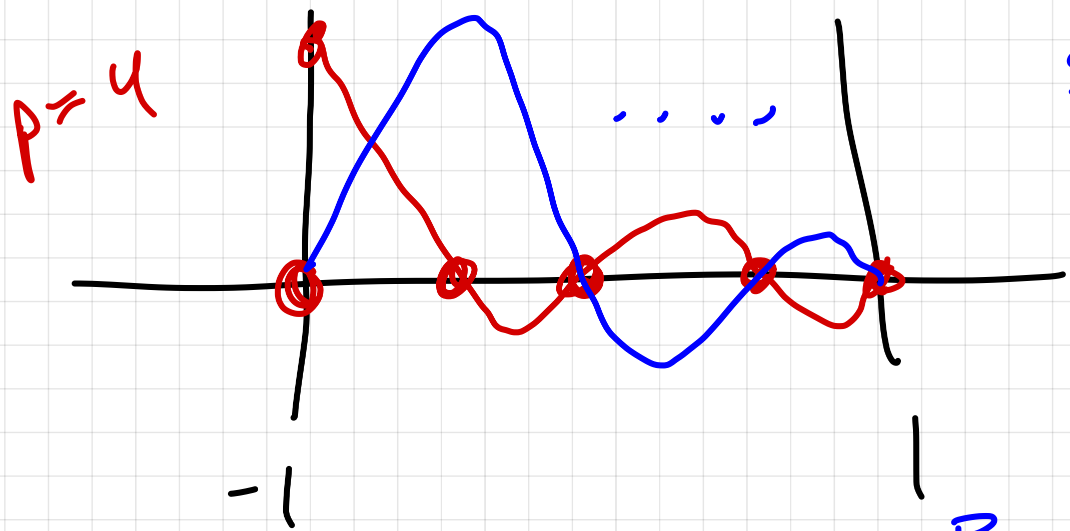
let $\{u_i\}_{i=0}^{n-1}$ be a basis for $\mathbb{R}^n$:

Find $\underline{x}$ st.

$$u_i^T (b - A\underline{x}) = 0 \quad \forall\ u_i,\ i=0,\dots,n-1$$

Take $[-1, 1]$ and fix "p".



5 basis functions

let $u(x) = \sum_{i=0}^{P} u_i \phi_i(x)$

Consider any other function $v(x)$.

Find u_i $i=0,\dots,P$ such that

$\|u(x) - v(x)\|_2 \rightarrow$ minimized

let $\int u(x)\,\phi(x) = \int v(x)\,\phi(x) \quad \forall\, \phi(x)$

Goal: given $g(x)$ $[-1,1]$
find $u(x)$ such that

$$u(x) = g(x)$$

Find $u \in P^p$ such that

$$u(x) = g(x)$$

Find $u \in P^p$ such that

$$\int_{-1}^{1} u \cdot v \, dx = \int g(x) \, v \, dx \quad \forall v \in P^p.$$

$$\int_{-1}^{1} (u-g) \, v \, dx = 0 \quad \forall v \in P^p$$

let $u = \sum_{i=0}^{P} u_i \phi_i(x)$

Find $\underline{u}$ s.t.

$$\int_{-1}^{1} \left(\sum_{i=0}^{P} u_i \phi_i(x) \right) v \, dx = \int_{-1}^{1} g(x) \, v(x) \, dx \quad \forall v \in P^P.$$

but $v = \sum c_i \phi_i(x)$

Find $\underline{u}$ s.t.

$$\int_{-1}^{1} \left(\sum_{i=0}^{P} u_i \phi_i(x) \right) \phi_j(x) \, dx = \int_{-1}^{1} g(x) \phi_j(x) \, dx$$

$$\sum_{i=0}^{P} u_i \int_{-1}^{1} \phi_i(x) \phi_j(x) \, dx = \int_{-1}^{1} g(x) \phi_j(x)$$

Let $M_{ji} = \int_{-1}^{1} \phi_i(x) \phi_j(x) \, dx$

$$\underline{M}\ \underline{u} = \underline{G}.$$

$$\underline{G} = \int_{-1}^{1} g(x)\, \phi_j(x)\, dx$$

Q Let $M_{ji} = \int_{-1}^{1} \phi_i(x)\, \phi_j(x)\, dx$

$i, j = 0, \dots, p$

let $\underline{u}$ = vector of p+1 values.

what does $\underline{u}^T M \underline{u}$ represent?

given any g.

Find $u \in V$ such that

$$\|u-g\| \to \min.$$