Today 4/5

- Intro to theory
- Work on outlines
- Talk on talks (maybe Monday)

let Ω be open

let $f \in H^1(\Omega)$

$$-\nabla \cdot \nabla u + u = f$$

$u(\underline{x}) = 0 \quad \underline{x} \in \partial\Omega$

let $V = H^1_0(\Omega)$

IBP

$$\int_\Omega -\nabla \cdot \nabla u \, v + uv = \int_\Omega fv$$

$$\int_\Omega \nabla u \cdot \nabla v + uv - \underbrace{\int_{\partial\Omega} \hat{n} \cdot \nabla u \, v}_{=0 \text{ since } v \in H^1_0(\Omega)} = \int_\Omega fv$$

Int. by parts

$$\underbrace{\int_\Omega \nabla \cdot (a\underline{b}) = \int_{\partial\Omega} \underline{n} \cdot (a\underline{b})}$$

$$= \int_\Omega \nabla a \cdot \underline{b} + \int_\Omega a \nabla \cdot \underline{b}$$

Find $u \in H_0^1(\Omega)$ s.t.

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v + uv}_{\langle \nabla u, \nabla v \rangle + \langle u, v \rangle} = \underbrace{\int_\Omega f v}_{\langle f, v \rangle} \quad \forall v \in V$$

L^2-inner prod.

$$\langle \nabla u, \nabla v \rangle + \langle u, v \rangle = \langle f, v \rangle$$

bilinear form

$$a(u,v) = g(v)$$

linear functional

Let $(V, \|\cdot\|_V)$ be a Banach space

A linear functional is a linear function

$$g : V \to \mathbb{R}$$

ex: $g(v) = \int_\Omega v \, dx$

A linear functional is bounded if
a.k.a.
continuous

$$|g(v)| \leq c \cdot \|v\| \quad \forall v \in V$$

dual space

let V' = space of all bounded linear functionals on V,

w. norm

$$\|g\|_{V'} = \sup_{\substack{v \in V \\ v \neq 0}} \frac{|g(v)|}{\|v\|_V}$$

Back to the problem:

$$g(v) = \langle f, v \rangle$$ ← L^2- inner prod.

is $g(\cdot)$ a b.l.f. ?

$$\|g\|_{V'}^2 = \sup_{\substack{v \in V \\ v \neq 0}} \frac{|g(v)|^2}{\|v\|_V^2} \quad \forall f \in L^2$$

$$\leq \sup_{v \in V} \frac{|\langle f, v \rangle|^2}{\|v\|_V^2}$$

V ← H^1

$$\leq \frac{\|f\|^2 \|v\|^2}{\|v\|^2 + \|Dv\|^2} \quad \forall v \in V$$

D ← derivative

$$\leq \frac{\|f\|^2 \|v\|^2}{\|v\|^2}$$

$$= \|f\|^2$$

Let $g(\cdot)$ be a bounded linear functional (b.l.f.)

Then there exists a unique! $u \in V$ s.t.

$$g(v) = (u,v)_V \quad \forall\, v \in V$$

Riesz Representation Theorem.

Linear Algebra

let $V = \mathbb{R}^n$

let $g : V \to \mathbb{R}$ be a linear fct!

$\underline{v} \in V$ is written as

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + \cdots v_n \underline{e}_n$$

$$\to g(\underline{v}) = v_1 g(\underline{e}_1) + v_2 g(\underline{e}_2) + \cdots v_n g(\underline{e}_n)$$

let $w_i = g(e_i)$

$$= \langle \underline{v}, \underline{w} \rangle$$

$$g(v) = \int f\, v \, dx$$

$$= \int \cos(x)\, v$$

RRT

Proof-ish

Let $w \in \underbrace{N(g)}_{\text{Null space of } g(\cdot)}{}^{\perp}$ ← all vectors orthogonal to all of $N(g)$

$$R(g) = N(g) \oplus N(g)^{\perp}$$

Let $\alpha := g(w) \neq 0$

Pick any $v \in V$.

$$g(v) = \frac{g(w)}{\alpha} \cdot g(v)$$

$$= g\left(\underbrace{\frac{g(v)}{\alpha} \cdot w}\right)$$

$$\rightarrow \quad g\left(v - \frac{g(v)}{\alpha} w\right) = 0$$

let $z = v - \frac{g(v)}{\alpha} \cdot w$ (Gram-Schmidt ??)

$\to z \in N(g)$ since $g(z) = 0$

$(z, w)_V = 0$

$\to \left(v - \frac{g(v)}{\alpha} w, w\right)_V = 0$

$\to \frac{g(v)}{\alpha} (w, w)_V = (v, w)_V$

put here

$\to g(v) = \Big(v, \underbrace{\frac{\alpha}{(w,w)_V} w}_{u}\Big)_V$

$= (v, u)_V$

is it unique?

suppose we have two: $u, \hat{u}$

$$g(v) = (u, v)_V$$

and

$$g(v) = (\hat{u}, v)_V \qquad \forall v.$$

$$\rightarrow \quad (u - \hat{u}, v)_V = 0 \qquad \forall v, \; v \neq 0$$

$$\rightarrow \quad u - \hat{u} = 0 \qquad \text{by inner prod.}$$

$(w, w) \geq 0$

$(w, w) = 0$ iff $w = 0$.

back to the problem:

<u>ex</u> $\quad f = \cos(x)$

$$g(v) = \int_{\Omega} \cos(x)\, v$$

$$= \langle \cos(x), v \rangle$$

$(u-\hat{u}, v) = 0 \quad \forall\, v \in V.$

pick $v = u - \hat{u}$

then

$(u-\hat{u}, u-\hat{u}) = 0$

$\Rightarrow \quad u - \hat{u} = 0$ by inner prod.

Back to

$$\underbrace{\langle \nabla u, \nabla v\rangle + \langle u, v\rangle}_{a(u,v)} = \underbrace{\langle f, v\rangle}_{g(v)}$$

L^2

any f. in L^2

What do we know?

$$a(u,v) = \langle u,v\rangle + \langle \nabla u, \nabla v\rangle$$

$$= (u,v)_{H^1}$$

So given blf $g(\cdot)$

there exists unique u s.t.

$$g(v) = \langle u, v\rangle_{H^1}$$

RRT.

→ exists unique solution

What about

$$-\nabla \cdot \nabla u = f$$

$\Rightarrow$ Find $u \in H_0^1$ s.t.

$$\underbrace{\langle \nabla u, \nabla v \rangle}_{a(u,v)} = \underbrace{\langle f, v \rangle}_{g(v)} \quad \forall v \in V$$

New tool

let V be Hilbert

coercivity : $a(\cdot,\cdot): V \to \mathbb{R}$ is coercive if $\exists\ c_0 > 0$ such that

$$c_0 \|u\|_V^2 \le a(u,u) \quad \forall\, u \in V.$$

continuity: $a(\cdot,\cdot): V \to \mathbb{R}$ is continuous if $\exists\ c_1 > 0$ such that

$$|a(u,v)| \le c_1 \|u\|_V \|v\|_V \quad \forall\, u,v.$$

if both: then $a(\cdot,\cdot)$ is V-elliptic

Let $a(\cdot,\cdot)$ be V-elliptic.

Let $g(\cdot)$ be a b.l.f.

$\Rightarrow \exists!\ u \in V$ s.t. $a(u,v) = g(v) \quad \forall v$

(exist uniqe)

Lax-Milgram Thm.

Proof

- $a(\cdot,\cdot)$ defines an inner product
- R.R.T.