Today

- Existence/uniqueness
- W: approximations

Recall

Let V be a Hilbert space, $\langle \cdot , \cdot \rangle_V$

Let $g(\cdot)$ be a b.l.f.

$|g(v)| \leq c \cdot \|v\|_V.$

Then there exist a unique u

s.t. $g(v) = \langle u, v \rangle_V$ $\forall v \in V.$

Proving existence + uniqueness:
Three case

difficulty

1. if $-\nabla\cdot\nabla u + u$ → weak form

$a(u,v) = \langle u,v\rangle_{H^1}$

→ RRT

2. if $a(u,v)$ is coercive + continuous + symmetric

→ a is an inner prod. + RRT

3. if $a(u,v)$ is coercive + continuous

→ Lax-Milgram.

coercive: $\exists\, c_0 > 0$ s.t.

$$c_0 \|u\|_V^2 \le a(u,u) \quad \forall\, u \in V.$$

continuity: $\exists\, c_1 > 0$ s.t. $|a(u,v)| \le c_1 \|u\|_V \|v\|_V$

$\forall\, u,v \in V$

Theorem Lax-Milgram symmetric version

Assume $a(\cdot,\cdot)$ is coercive and contin. and symmetric.

Assume $g(\cdot)$ is a b.l.f.

The $\exists!$ $u \in V$ s.t.

$$a(u,v) = g(v) \quad \forall v \in V.$$

$\int_\Omega f(x)\, v \, dx$

Proof

Symm.

1. $a(u,v) = a(v,u)$

2. $a(cu+v, w) = c\,a(u,w) + a(v,w)$ bilinearity

3. $a(u,u) \geq c \cdot \|u\|_V^2 \geq 0$ coercivity

4. if $u \equiv 0$ a.e.

   then $a(u,u) \leq c_1 \|u\|_V^2 = 0$ continuity

5. if $a(u,u) = 0$

   then $0 \leq \|u\|_V^2 \leq a(u,u) = 0$ so $u \equiv 0$ a.e.

   coercivity

→ $a(\cdot,\cdot)$ is an inner prod.

→ RRT.

Model problem

$$-\nabla \cdot \nabla u = f$$
$$u = 0 \quad \text{on } \partial\Omega$$

→ Find $u \in V$ s.t.

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx}_{a(u,v)} = \underbrace{\int_\Omega f v \, dx}_{g(v)} \quad \forall v$$

pick this somehow

→ pick V

→ show coercivity + continuity

$$-\nabla \cdot \nabla u = f$$

$$n \cdot \nabla u = 0$$

$$-\nabla \cdot k(\underline{x}) \nabla u = f$$

$$k(x) > 0$$

$$-\Delta \underline{u} + \nabla p = f$$

$$\nabla \cdot \underline{u} = 0$$

$$-\nabla \cdot \nabla u = f$$

$$\downarrow$$

let $\underline{q} = \nabla u$

$$\rightarrow \begin{cases} -\nabla \cdot \underline{q} = f \\ \underline{q} - \nabla u = 0 \end{cases}$$

Introduce $\underline{v}$, w

$$\rightarrow \int_\Omega -\nabla \cdot \underline{q}\, w = \int_\Omega f w$$

$$\int_\Omega \underline{q} \cdot \underline{v} - \nabla u \cdot \underline{v} = 0$$

$$\forall\, \varphi, \underline{v} \in W \times V$$

$$-\nabla \cdot \nabla u = f$$

$$-\nabla \cdot k(x) \nabla u = f$$

$$\downarrow$$

Find u s.t.

$$\int k(x) \nabla u \cdot \nabla v = \int f v \, dx$$

$$a(u, v)$$

Theorem Lax-Milgram

Suppose $a(\cdot,\cdot)$ is a coercive + continuous bilinear form on V. Suppose $g(\cdot)$ is a b.l.f. on V.

Then $\exists!$ $u \in V$ s.t.

$$a(u,v) = g(v) \quad \forall v \in V.$$

Proof-ish

Take any $u \in V$.

let $a_u(v) := a(u,v)$

then a_u is b.l.f:

continuity

$$|a_u(v)| \le \underbrace{c \cdot \|u\|_V}_{C} \|v\|_V$$

$\Rightarrow$ by RRT $\exists!$ t_u such that

$$a_u(v) = \langle v, t_u \rangle_V \quad \forall v \in V.$$

We have a map!

$$T : V \to V$$

$$u \qquad t_u$$

Suppose T is "onto" V.

Let t_g be such that

$$g(v) = \langle v, t_g \rangle_V \quad \text{RRT} \quad \forall v \in V.$$

Then there is a $u \in V$ s.t.

$$t_g = Tu$$

So

$$\begin{aligned} g(v) &= \langle v, t_g \rangle_V \\ &= \langle v, Tu \rangle_V \\ &= a(u,v) \qquad \forall v \in V \end{aligned}$$

Suppose $t_g = T\hat{u}$ for another $\hat{u}$.

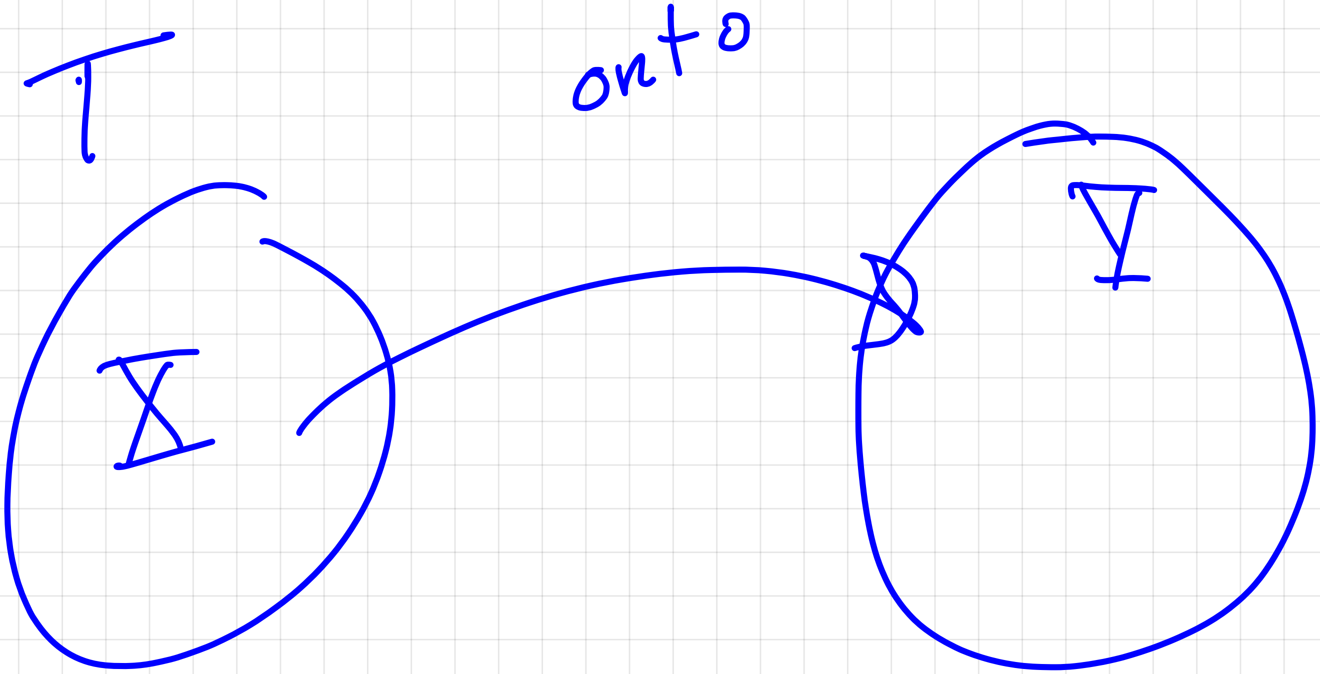
Then $a(u,v) = a(\hat{u},v)$

$\rightarrow \quad a(u-\hat{u},v) = 0 \quad \forall v.$

$\rightarrow \quad a(u-\hat{u}, u-\hat{u}) = 0$

$\rightarrow \quad u = \hat{u}$

onto



$\forall y \in \quad \exists x \in X$

s.t. $Tx = y$

let $a(u,v) = \int_{\Omega} \nabla u \cdot \nabla v$

$g(v) = \int_{\Omega} f\, v$

Lax-Milgram

if $a(\cdot,\cdot)$ is coercive + continuous on V.

and if $g(\cdot)$ is a b.l.f. on V.

Then $\exists!$ u s.t.

$a(u,v) = g(v)$ $\quad \int \nabla u \cdot \nabla v = \int f v$

$\forall v$.

Steps:

1) pick V. ← hard?
2) show $a(\cdot,\cdot)$ continuous on V. ← easy
3) show $a(\cdot,\cdot)$ is coercive on V. ← hard
4) show $g(\cdot)$ is a b.l.f. ← easy

1) Let $V = H_0^1(\Omega)$

2) $|a(u,v)|$ want $\leq c_1 \cdot \|u\|_{H^1} \cdot \|v\|_{H^1}$ $\forall\, u, v.$

$$= \left| \int_\Omega \nabla u \cdot \nabla v \, dx \right|$$

Hölder ineq.

$$\leq \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2} \left(\int_\Omega |\nabla v|^2 dx \right)^{1/2}$$

$$\leq \left(\int_\Omega \overset{\geq 0}{|u|^2} + |\nabla u|^2 dx \right)^{1/2} \left(\int_\Omega \overset{\geq 0}{|v|^2} + |\nabla v|^2 dx \right)^{1/2}$$

$$\leq \|u\|_{H^1} \|v\|_{H^1}$$

3) Tool: $\boxed{\begin{array}{l}\text{let } \Omega \text{ be bounded} \\ \text{let } u \in H_0^1 \\ \text{Then there is } c>0 \text{ such that} \\ \qquad \|u\|_{L^2} \le c \cdot \|\nabla u\|_{L^2}\end{array}}$

Poincaré - Friedrichs ineq.

$$a(u,u) = \int_\Omega \nabla u \cdot \nabla u \, dx$$

$$= \|\nabla u\|_0^2$$

$$\ge \frac{1}{c^2+1} \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right)$$

4) for some $f(x)$ show $g(\cdot)$ is bounded.

$$g(v) = \int_{\Omega} f(x)\, v \, dx$$

$$|g(v)| \leq \|f\|_{L^2} \|v\|_{L^2}$$