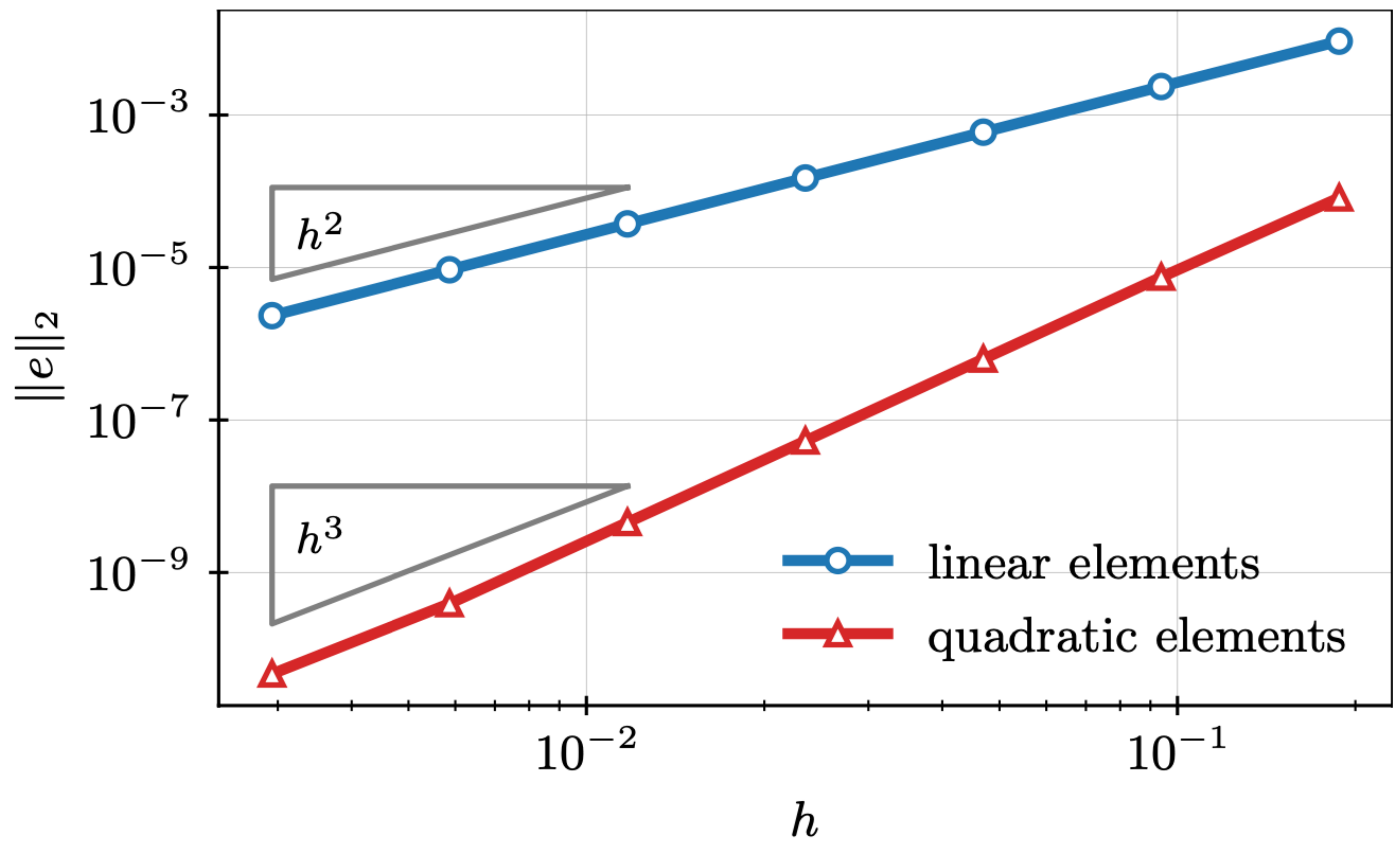
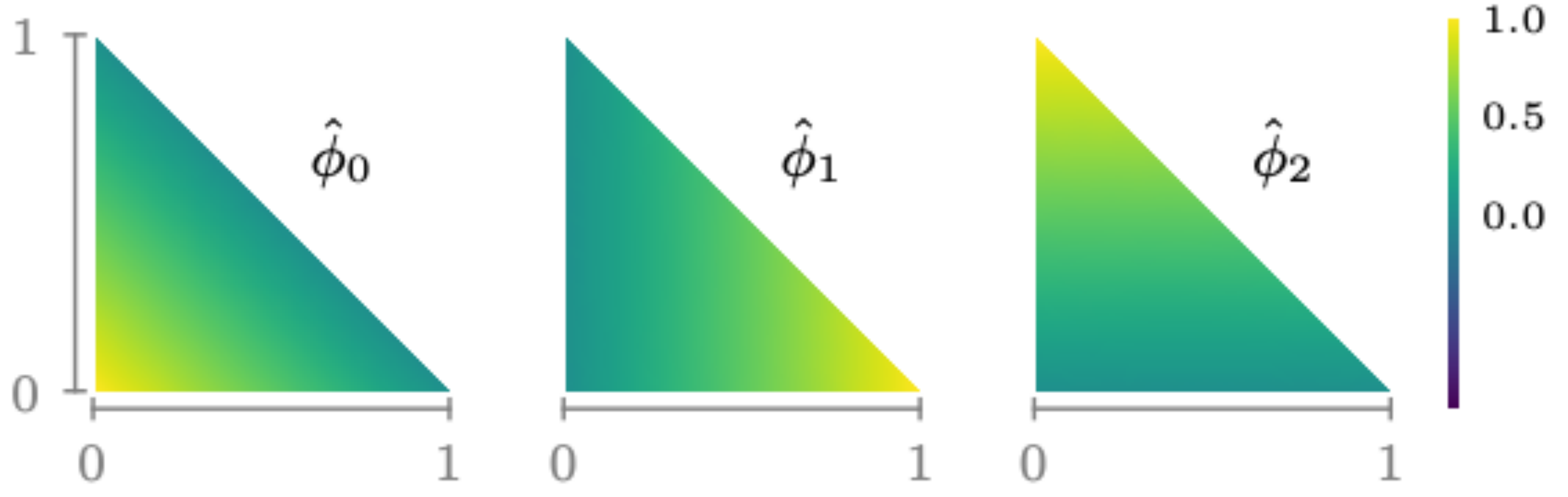
Lecture Plan

- Motivation
- Elements and Approximation Properties
- HW hints





Motivation

Solve PDEs numerically

$$\nabla \cdot \nabla u + u = f(x) \quad \text{for} \quad x \in \Omega$$
$$\text{with} \;\; u = 0 \;\; \text{on} \;\; \partial\Omega$$
$$f \in L^2(\Omega)$$

Strong form requiring a twice differentiable solution

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Strong form requiring a twice differentiable solution

$$\int_\Omega \nabla u \cdot \nabla v \; dx + \int_\Omega uv \; dx = \int_\Omega fv \; dx$$
$$\text{for all} \quad v \in V = H_0^1(\Omega)$$

Weak form requiring a once differentiable solution

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Weak form requiring a once differentiable solution

Existence and uniqueness of a weak solution

- Riesz Representation Theorem
- Lax-Milgram Theorem

Motivation

Solve PDEs numerically

$$\int_\Omega \nabla u \cdot \nabla v \; dx + \int_\Omega uv \; dx = \int_\Omega fv \; dx$$

$$\text{for all} \quad v \in V = H_0^1(\Omega)$$

Infinite dimensional subspace consisting of functions with continuous first derivative and zero on the boundary

$$u_h \in \mathcal{V}^h \subset \mathcal{V}$$

The goal of this lecture is to construct $\mathcal{V}^h$ such that it accurately represents $\mathcal{V}$

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Types of Elements and Approximation Properties

Section 8.3

Motivating questions

- How accurate is a solution from a finite-dimensional subspace ($\mathcal{V}^h \subset \mathcal{V}$)?
  - How do we choose $\mathcal{V}^h$?
- What are the bounds on the error ($u - u_h$)?
  - $\min_{u_h \in \mathcal{V}_h} ||u - u_h||_{\mathcal{V}}$

Types of Elements and Approximation Properties

How accurate is a solution from a finite-dimensional subspace?

Weak form

$$\int_\Omega \nabla u \cdot \nabla v \; dx + \int_\Omega uv \; dx = \int_\Omega fv \; dx$$

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Bilinear+Linear forms

$$a(u, v) = g(v)$$

$$\text{for all} \;\; v \in \mathcal{V}$$

Types of Elements and Approximation Properties

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Bilinear+Linear forms

$$a(u, v) = g(v)$$

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Ritz-Galerkin Approximation

$$a(u^h, v^h) = g(v^h)$$

$$\text{for all} \;\; v^h \in \mathcal{V}^h$$

Types of Elements and Approximation Properties

How accurate is a solution from a finite-dimensional subspace?

$$a(u, v) = g(v) \quad \text{for all} \quad v \in \mathcal{V}$$

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We would like to understand the relationship between $u \in \mathcal{V}$ and $u^h \in \mathcal{V}^h$

Types of Elements and Approximation Properties

How accurate is a solution from a finite-dimensional subspace?

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We would like to understand the relationship between $u \in \mathcal{V}$ and $u^h \in \mathcal{V}^h$

Orthogonality Relationship (Lemma 4.6)

Proof. Since u_h comes from the Ritz-Galerkin approximation, it must satisfy $a(u_h, v) = \langle f, v\rangle, \forall v \in \mathcal{V}^h$. Similarly, since u is the solution of the weak form, it must satisfy $a(u, v) = \langle f, v\rangle, \forall v \in \mathcal{V}$. Noting that $\mathcal{V}^h \subset \mathcal{V}$, this gives $a(u, v) = \langle f, v\rangle, \forall v \in \mathcal{V}^h$. Thus,

$$a(u - u_h, v) = a(u, v) - a(u_h, v) = \langle f, v\rangle - \langle f, v\rangle = 0 \quad \forall v \in \mathcal{V}^h.$$

Types of Elements and Approximation Properties

How accurate is a solution from a finite-dimensional subspace?

Orthogonality Relationship (Lemma 4.6)

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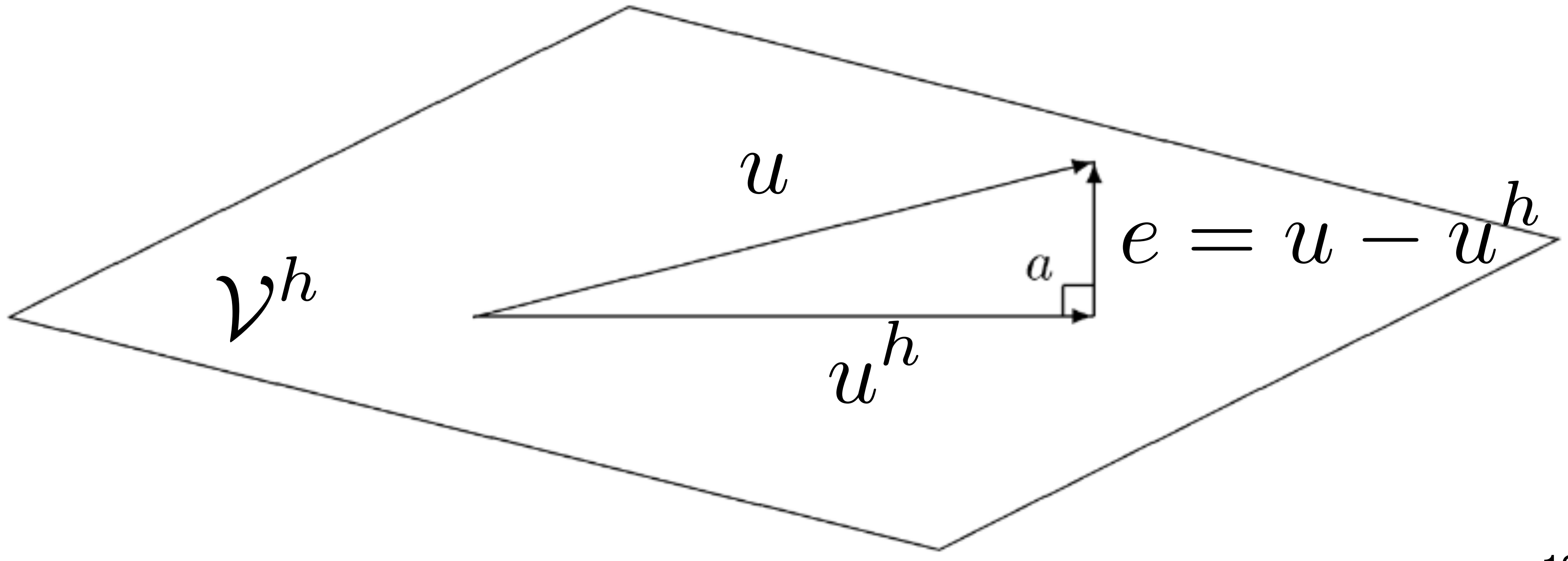


Relationship between solution to weak form and Ritz-Galerkin approximation

Types of Elements and Approximation Properties

How accurate is a solution from a finite-dimensional subspace?

Using the orthogonality property, we can show that the Ritz-Galerkin approximation generates the “best approximation” in the subspace

Lemma 8.28: Céa’s lemma

Let $\mathcal{V} \subset \mathcal{H}$ be a closed subspace of Hilbert space $\mathcal{H}$. Let $a(\cdot,\cdot)$ be a coercive and continuous bilinear form on $\mathcal{V}$. In addition, for a bounded linear functional, $g(\cdot)$, on $\mathcal{V}$, let $u \in \mathcal{V}$ satisfy

$$a(u, v) = g(v) \text{ for all } v \in \mathcal{V}. \tag{8.62}$$

Consider the finite-dimensional subspace $\mathcal{V}^h \subset \mathcal{V}$ and $u^h \in \mathcal{V}^h$ that satisfies

$$a(u^h, v^h) = g(v^h) \text{ for all } v^h \in \mathcal{V}^h. \tag{8.63}$$

Then,

$$\|u - u^h\|_{\mathcal{V}} \leq \frac{c_1}{c_0} \min_{v^h \in \mathcal{V}^h} \|u - v^h\|_{\mathcal{V}}, \tag{8.64}$$

where c_0 and c_1 are the coercivity and continuity constants for $a(\cdot,\cdot)$, respectively.

Types of Elements and Approximation Properties

How accurate is a solution from a finite-dimensional subspace?

Prove that $\|u - u^h\|_{\mathcal{V}} \leq \frac{c_1}{c_0} \min_{v^h \in \mathcal{V}^h} \|u - v^h\|_{\mathcal{V}}$

$$
\begin{aligned}
c_0\|u - u^h\|_{\mathcal{V}}^2 &\leq a(u - u^h, u - u^h) \quad \text{by coercivity} \\
&= a(u - u^h, u - v^h) + a(u - u^h, v^h - u^h) \\
&= a(u - u^h, u - v^h) \quad \text{since } v^h - u^h \in \mathcal{V}^h \\
&\leq c_1\|u - u^h\|_{\mathcal{V}}\|u - v^h\|_{\mathcal{V}} \quad \text{by continuity,}
\end{aligned}
$$

$c_0\|u\|_{\mathcal{V}}^2 \leq a(u, u)$ *for all* $u \in \mathcal{V}$

added $v^h - v^h$

$|a(u, v)| \leq c_1\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$

Shows how the solution of the weak form, u, and

Ritz-Galerkin approximation, u^h, are related

Types of Elements and Approximation Properties

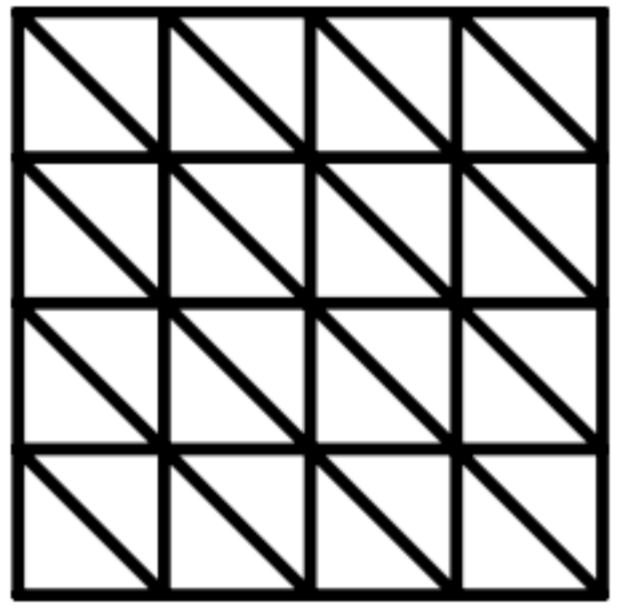
How accurate is a solution from a finite-dimensional subspace?

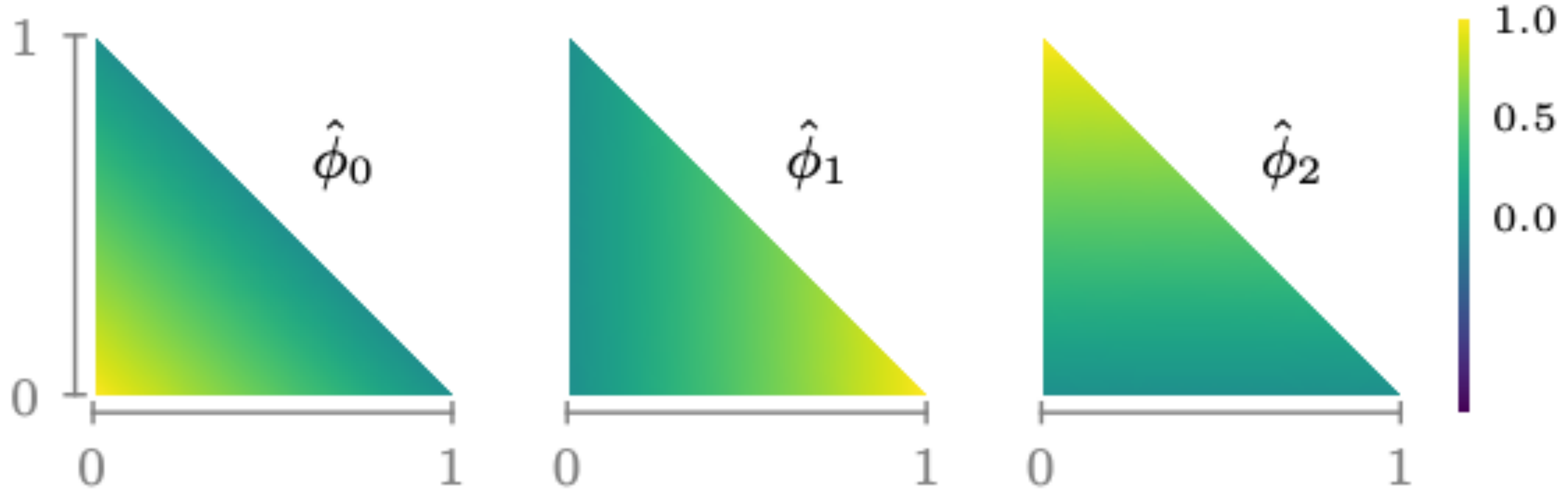
Cea's Lemma

$$\|u - u^h\|_{\mathcal{V}} \leq \frac{c_1}{c_0} \min_{v^h \in \mathcal{V}^h} \|u - v^h\|_{\mathcal{V}}$$

- Given that the $\frac{c_1}{c_0}$ is not unacceptably large, how do we construct/find $\mathcal{V}^h$?
- How large is the right-hand side?

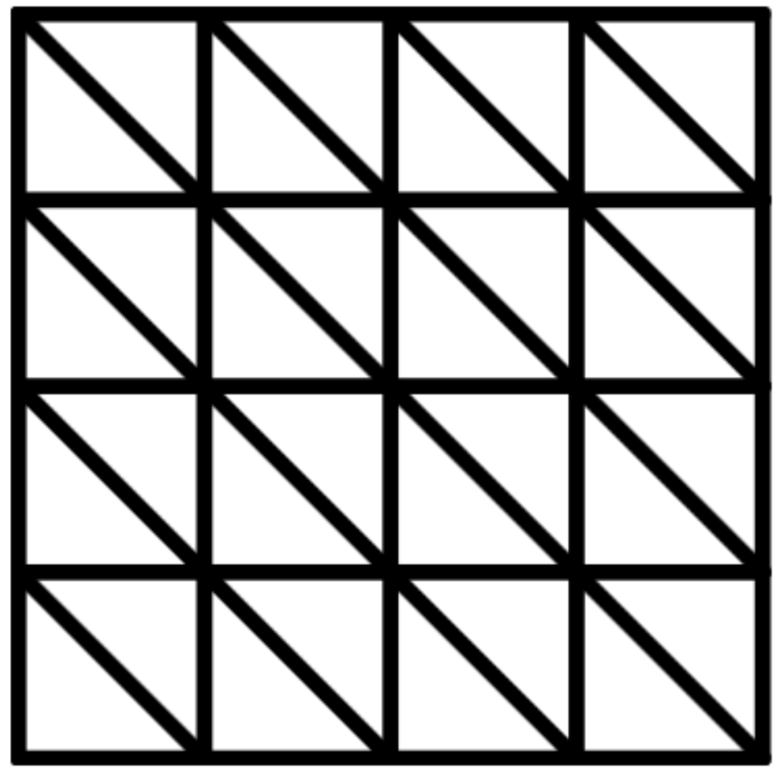
It depends on the mesh chosen and the piecewise-polynomial approximation spaces over that mesh.



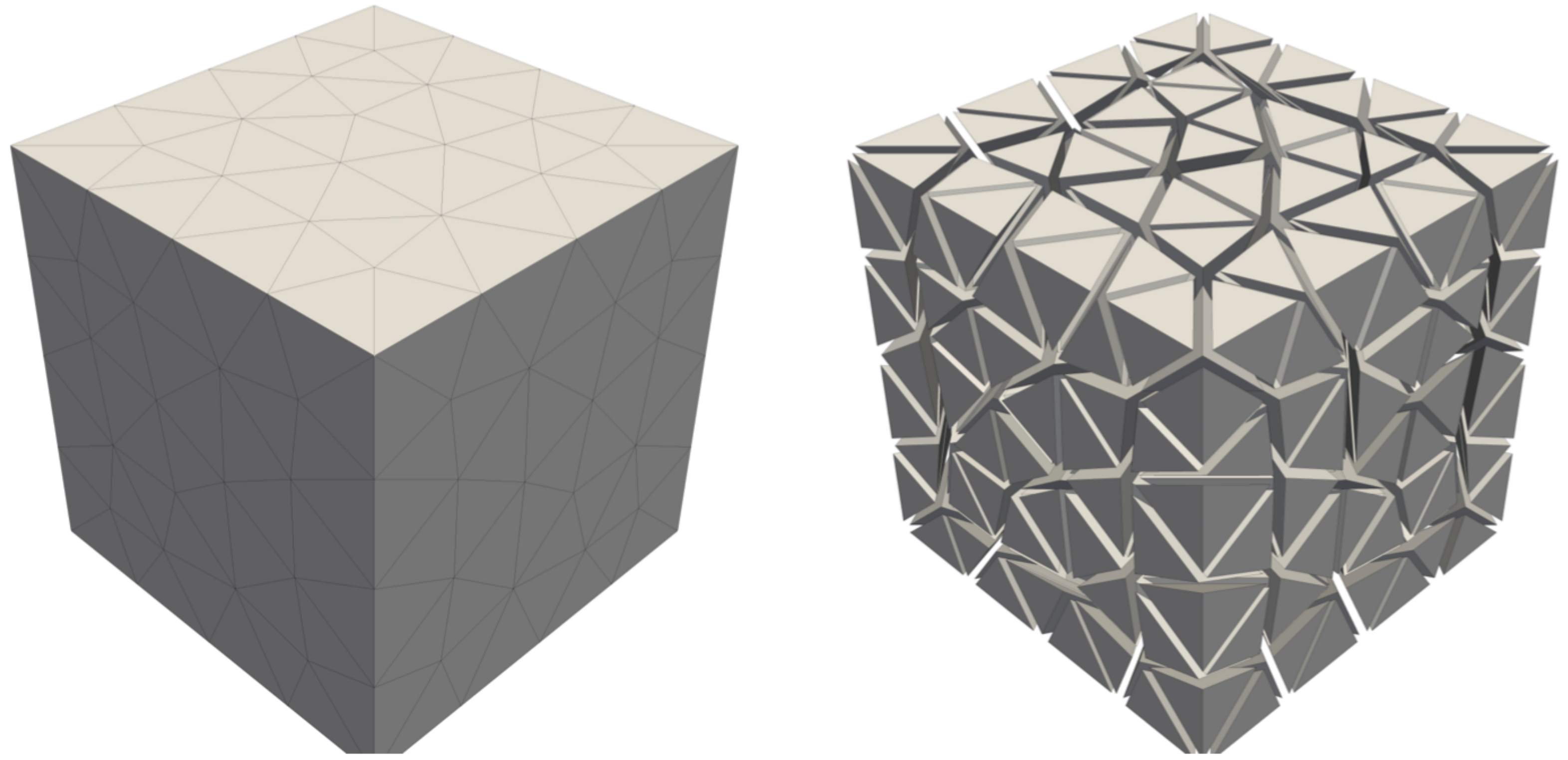


Types of Elements and Approximation Properties

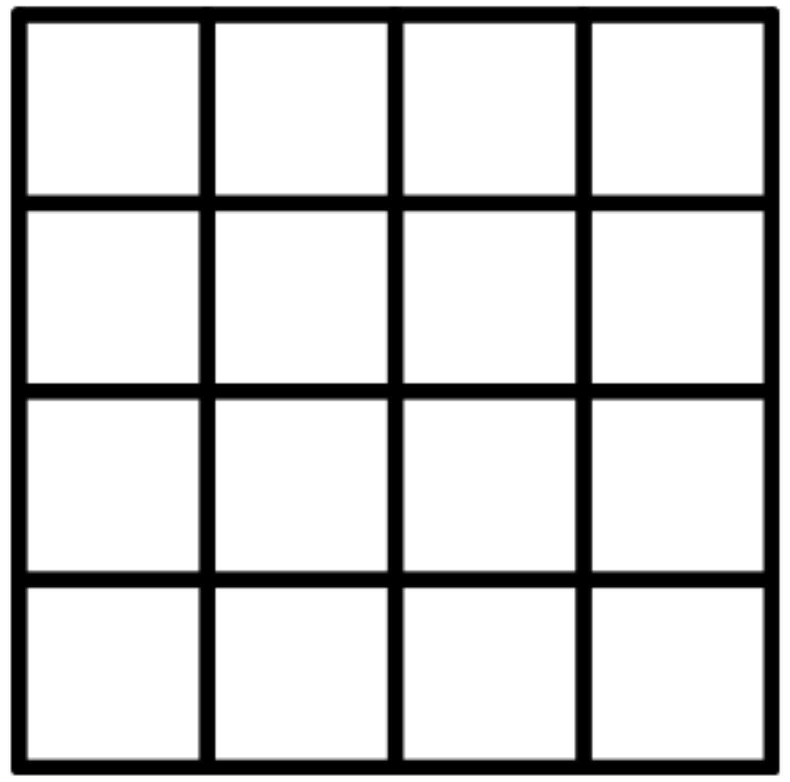
Meshes



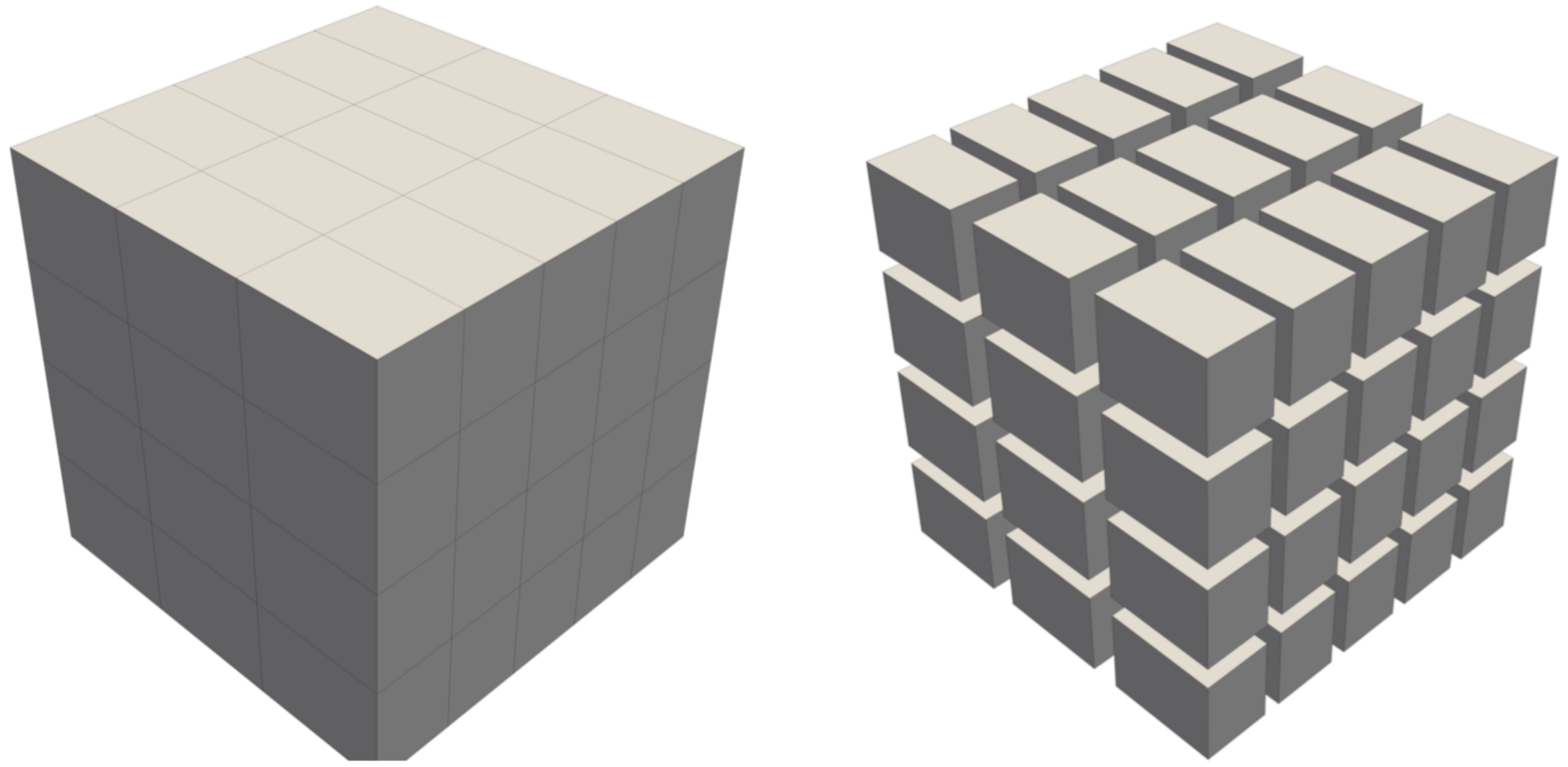
a. 2D triangular mesh



b. 3D tetrahedral mesh



a. 2D quadrilateral mesh

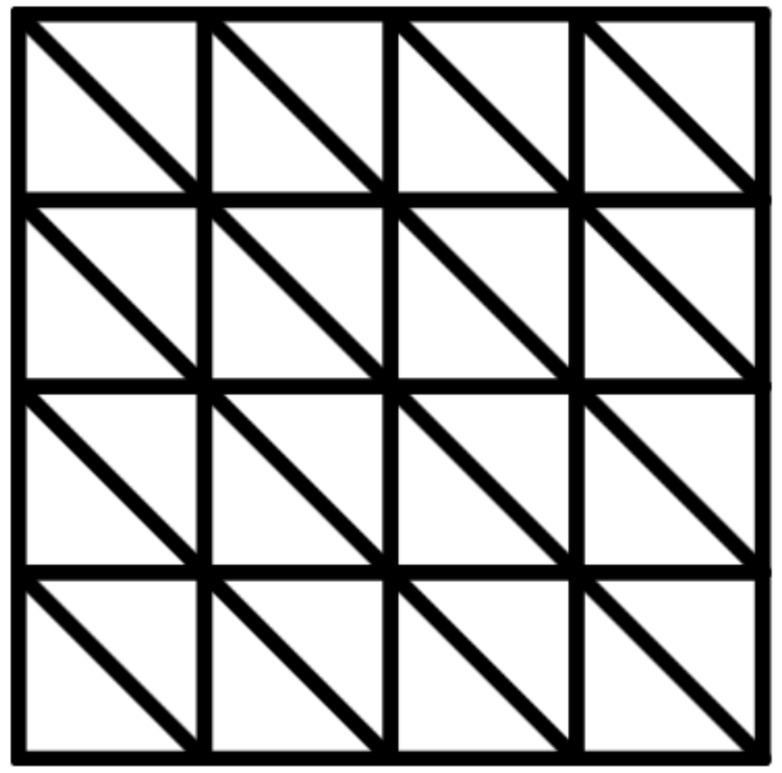


b. 3D hexahedral mesh

Types of Elements and Approximation Properties

Meshes

Triangular and Tetrahedral Elements, P_k



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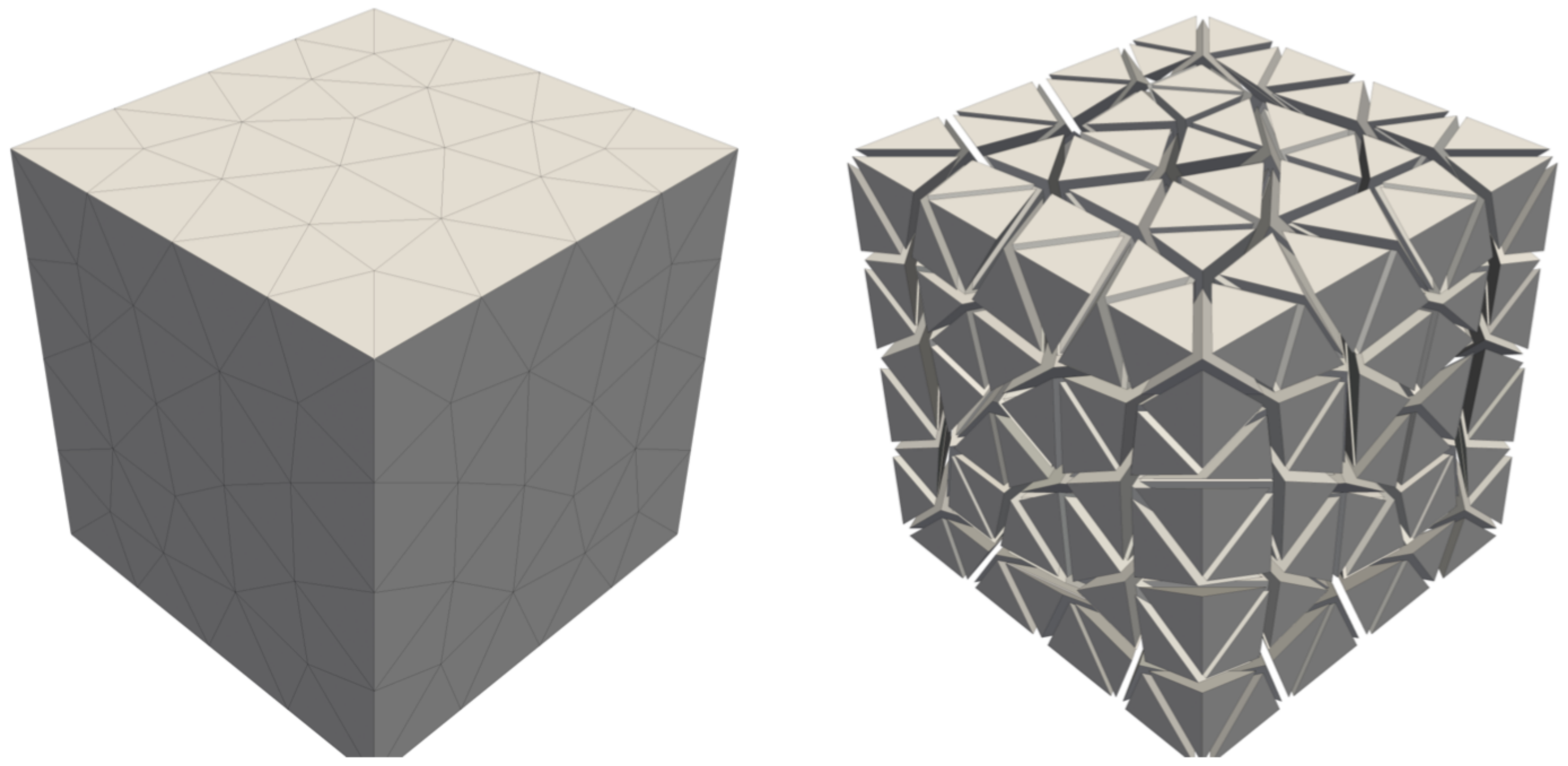
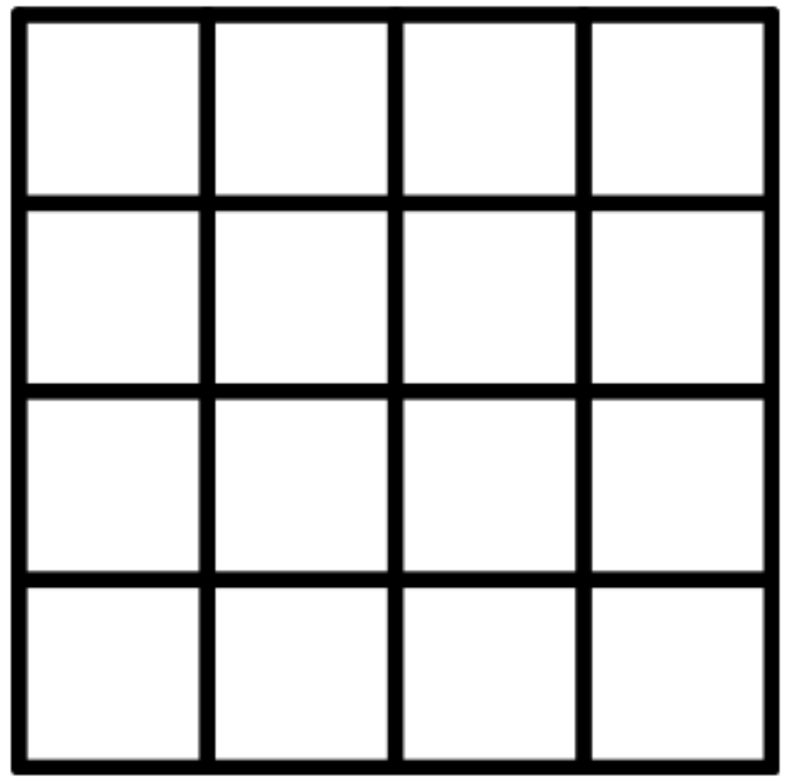
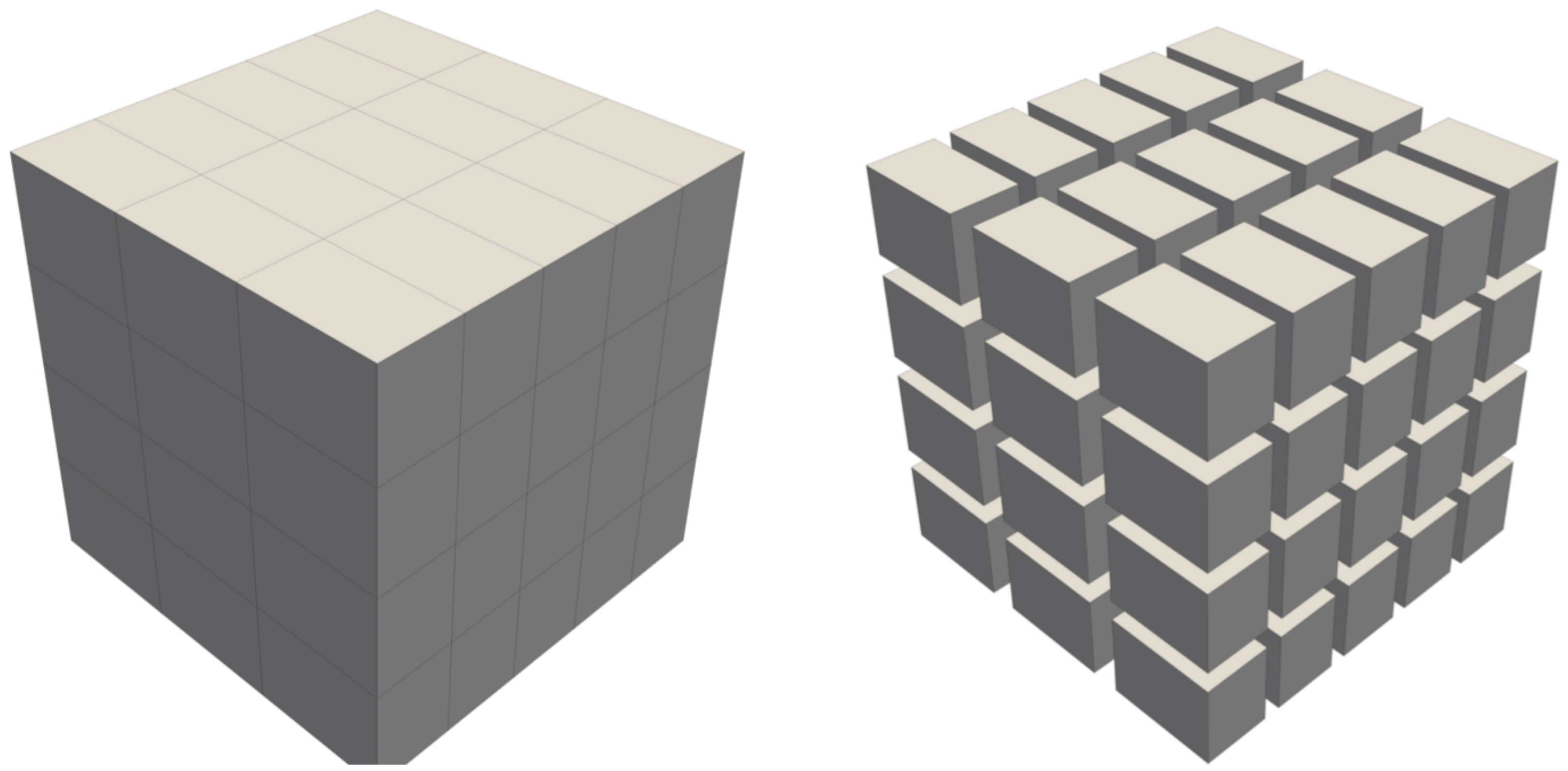


b. 3D tetrahedral mesh

Quadrilateral and hexahedral elements, Q_k



a. 2D quadrilateral mesh



b. 3D hexahedral mesh

Types of Elements and Approximation Properties

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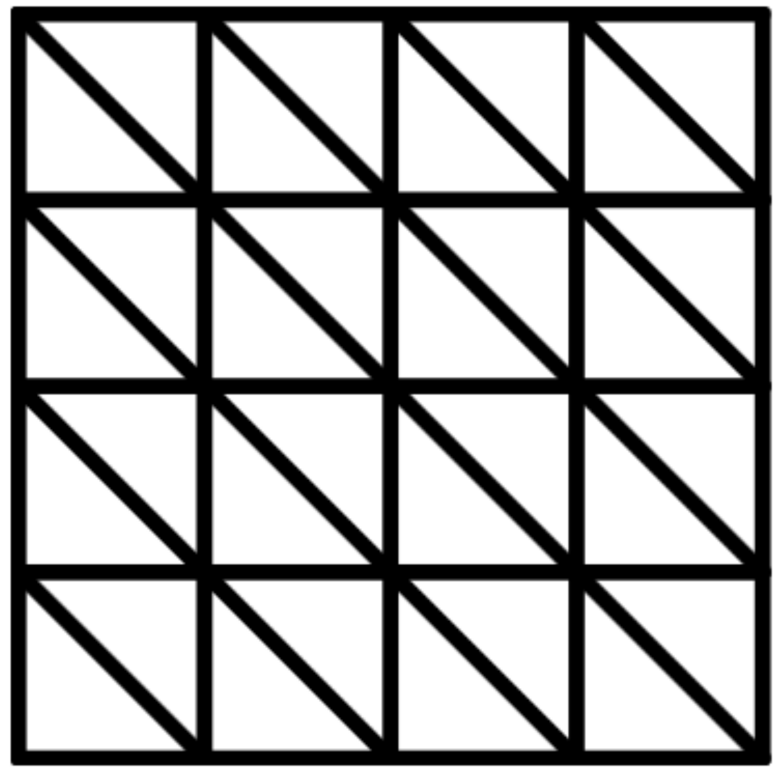
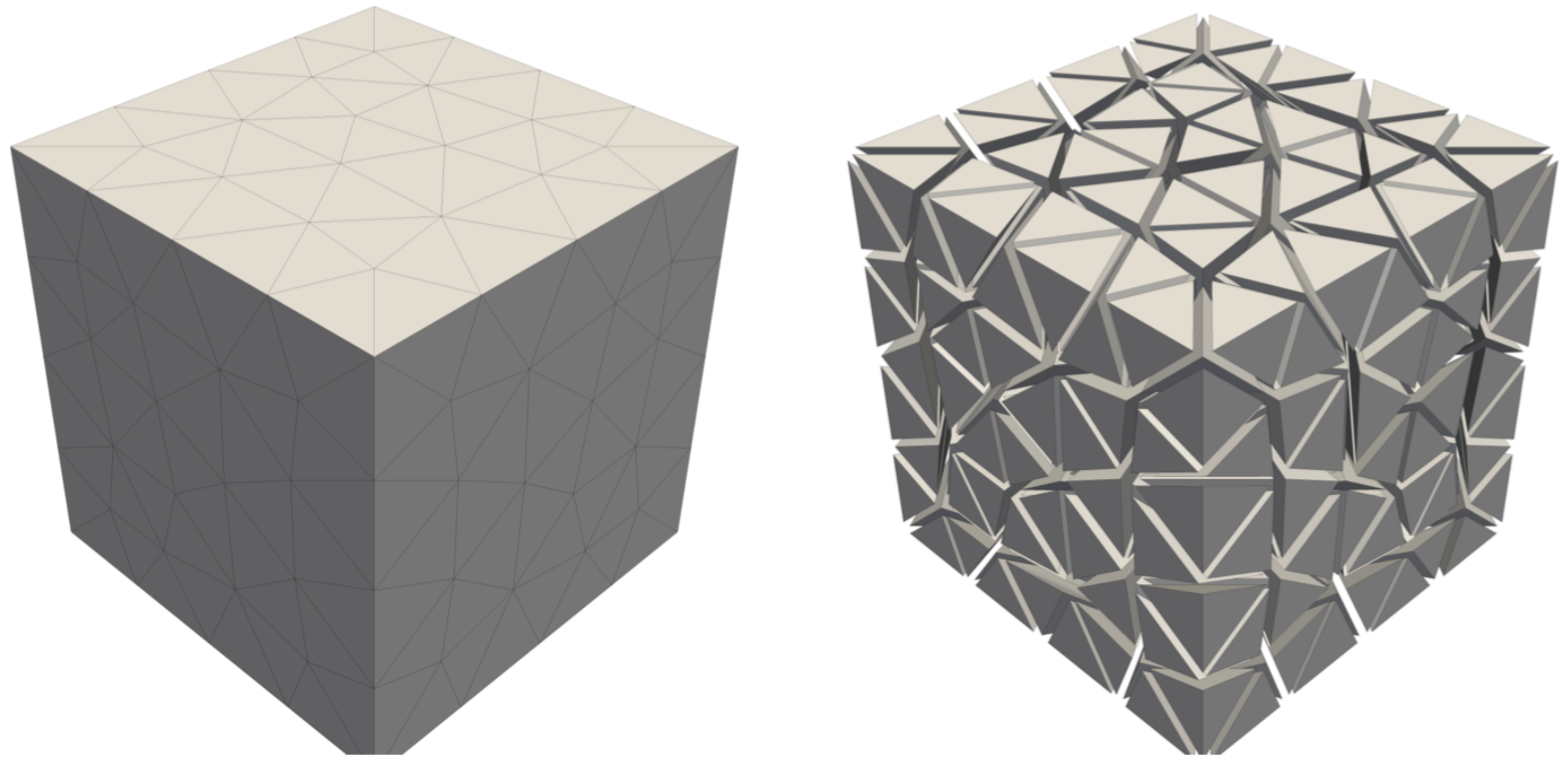
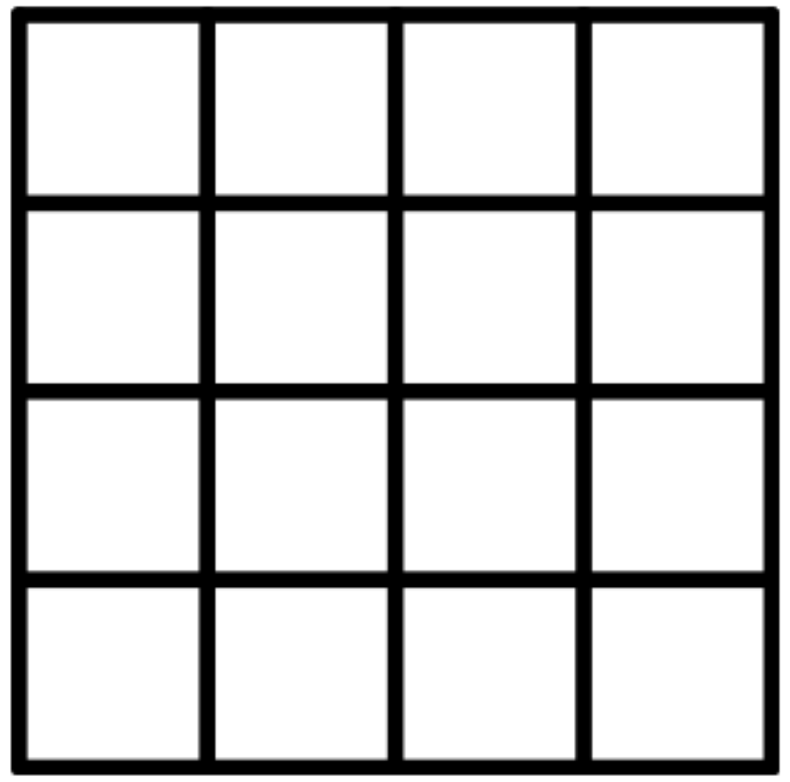


a. 2D triangular mesh



b. 3D tetrahedral mesh

Quadrilateral and hexahedral elements, Q_k



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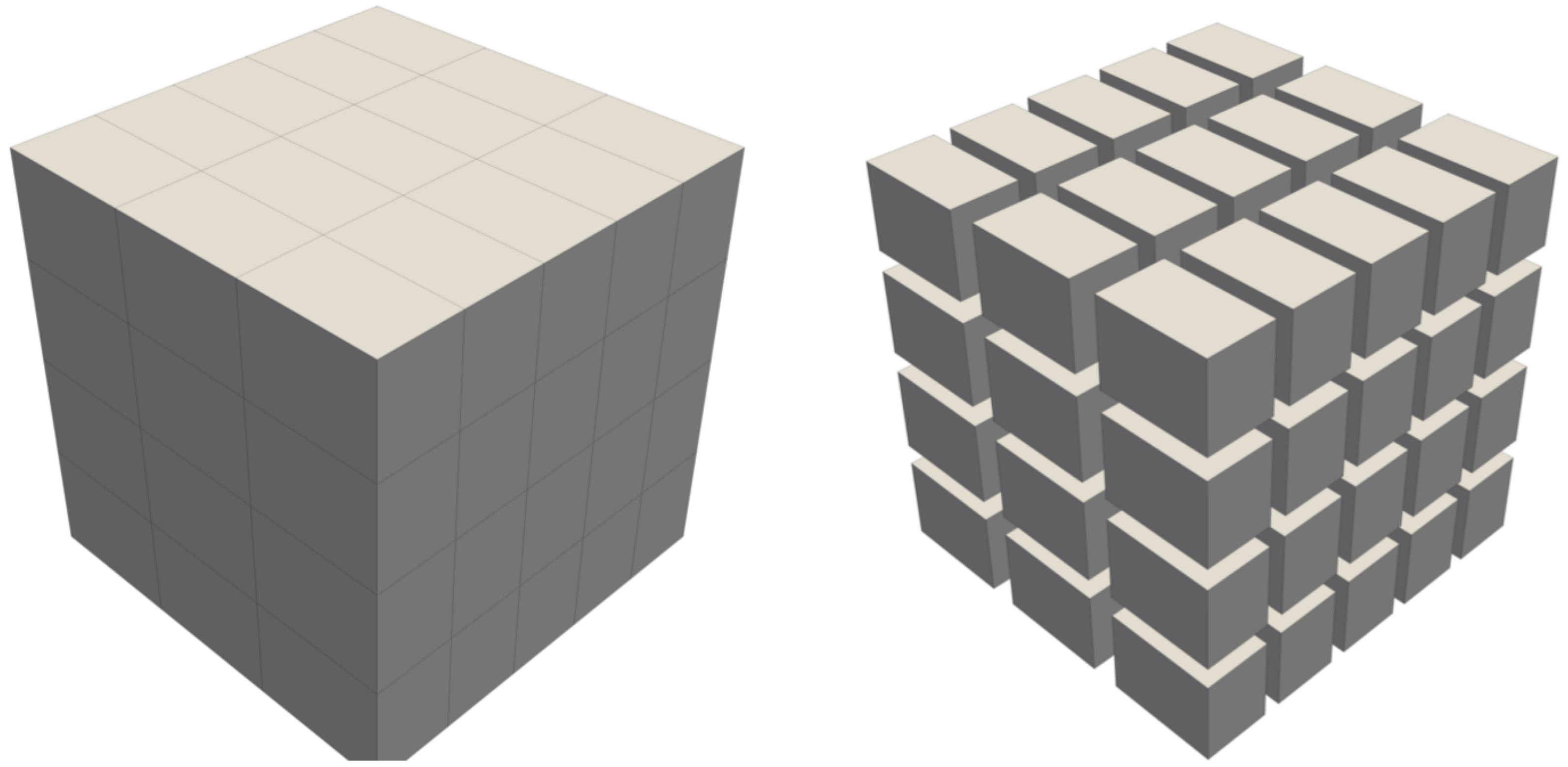


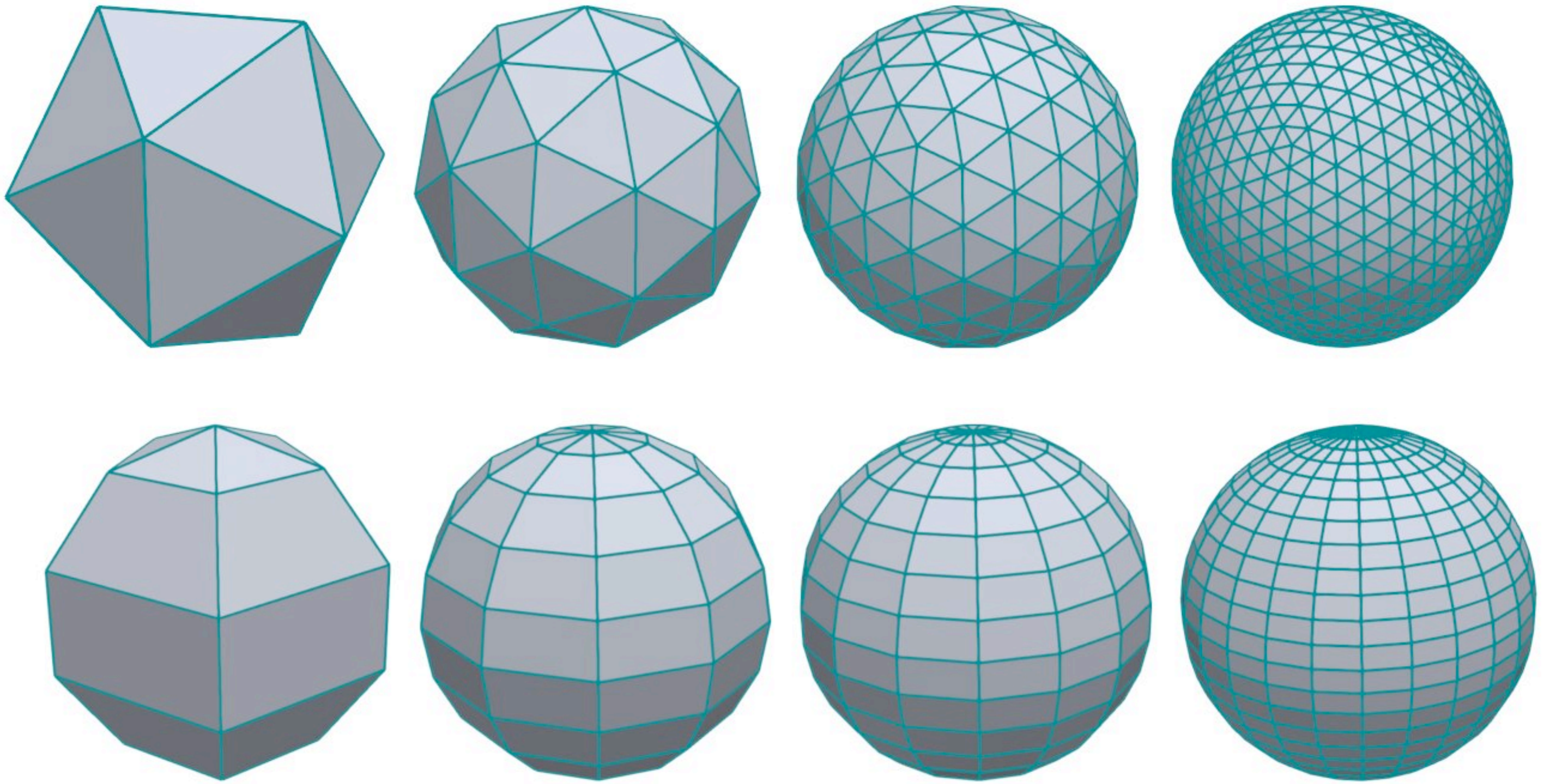
b. 3D hexahedral mesh

Types of Elements and Approximation Properties

Given a sphere domain Ω subdivide it into elements forming a Ω^h

P_k

Q_k

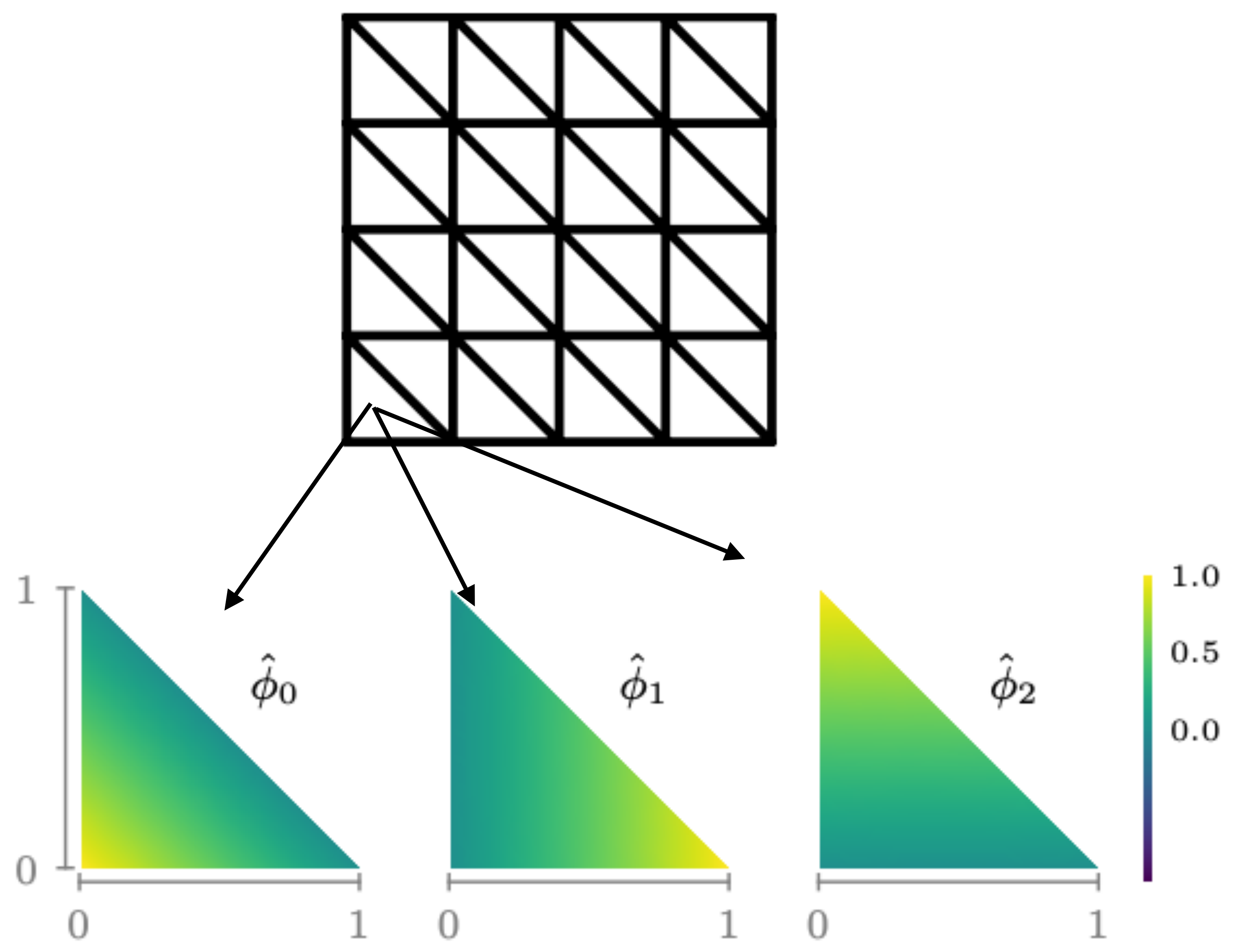


https://www.danielsieger.com/blog/2021/03/27/generating-spheres.html

Types of Elements and Approximation Properties

Triangular meshes on conforming polynomial spaces

- A triangular mesh, $\Omega^h = \{\tau\}$, is a set of triangles
- Once Ω^h is chosen, we can set an approximation space by defining
  - representations of functions over each τ
  - rules for how functions on one element relate to those on their neighbors (e.g. imposing continuity)



Types of Elements and Approximation Properties

Triangular meshes on conforming polynomial spaces

Continuous polynomial spaces are called **conforming** and discontinuous are **non-conforming**

$$P_k(\Omega^h) = \left\{ v \in C^0(\Omega^h) \;\middle|\; \forall \tau \in \Omega^h, v(\boldsymbol{x}) \text{ is a polynomial of degree no more than } k \text{ on } \tau \right\}$$

Types of Elements and Approximation Properties

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Examples:

- $P_1(\Omega^h)$: piece-wise linear function over triangular elements
- $P_2(\Omega^h)$: piece-wise quadratic function over triangular elements

Types of Elements and Approximation Properties

Triangular meshes on conforming polynomial spaces

Example 8.30: Piecewise Linears ($P_1(\Omega^h)$)

Take a triangle, τ, with three nodes, (x_1, y_1), (x_2, y_2), and (x_3, y_3), as shown in Figure 8.3.2. We define three basis functions over τ, written as

$$\phi_i(x, y) = a_i + b_i x + c_i y,$$

for $i = 1, 2, 3$, with the property that $\phi_i(x_j, y_j) = 1$ if $i = j$ and $\phi_i(x_j, y_j) = 0$ if $i \neq j$ for $j = 1, 2, 3$. Writing this out, we have, for $i = 1$,

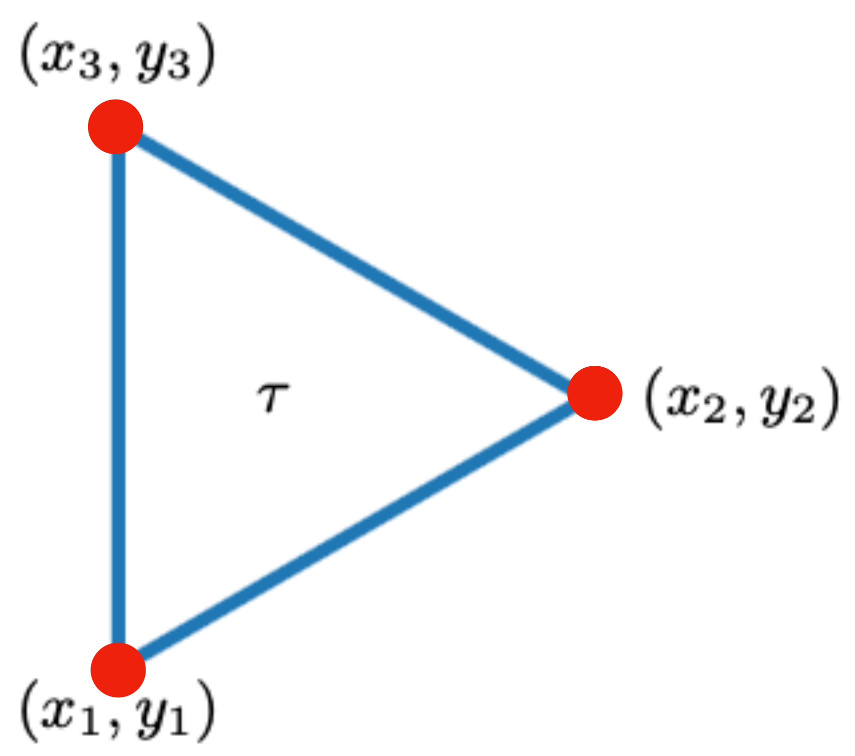
$$\begin{aligned} \phi_1(x_1, y_1) &= a_1 + b_1 x_1 + c_1 y_1 = 1 \\ \phi_1(x_2, y_2) &= a_1 + b_1 x_2 + c_1 y_2 = 0 \\ \phi_1(x_3, y_3) &= a_1 + b_1 x_3 + c_1 y_3 = 0, \end{aligned}$$

which leads to the linear system to be solved for a_1, b_1, c_1 as

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We know that the system will be uniquely solvable so long as the matrix is nonsingular, which occurs when its determinant is nonzero. From direct calculation,

$$\det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \det \begin{bmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x_3 - x_1 & y_3 - y_1 \end{bmatrix} = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1).$$



Types of Elements and Approximation Properties

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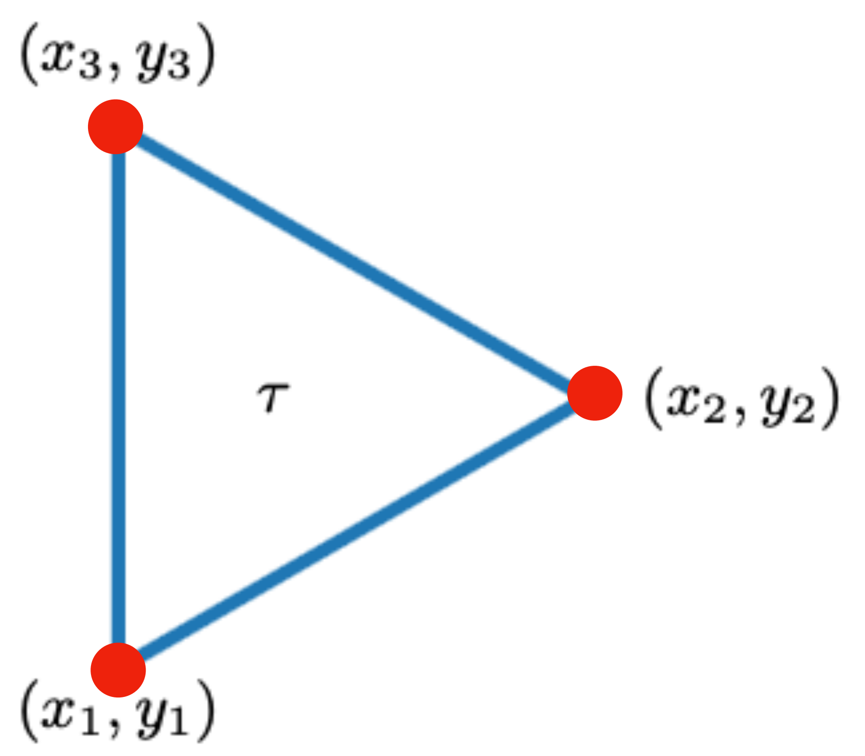
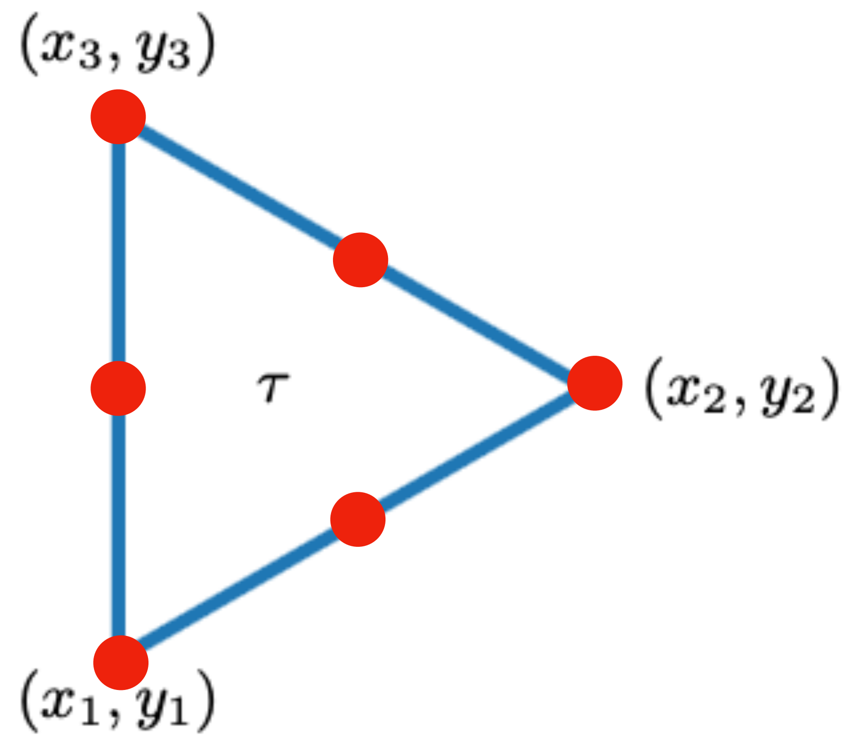


$$\begin{aligned} a_1 &= \frac{x_2 y_3 - y_2 x_3}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)},\\ b_1 &= \frac{y_2 - y_3}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)},\\ c_1 &= \frac{x_3 - x_2}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}.\end{aligned}$$

Types of Elements and Approximation Properties

Triangular meshes on conforming polynomial spaces

HW#4: do the previous calculation for $P_2(\Omega^h)$



$$\phi_i(x, y) = a_i + b_i\, x + c_i\, y + d_i\, x^2 + e_i\, xy + f_i\, y^2$$

Types of Elements and Approximation Properties

Triangular meshes on conforming polynomial spaces

Theorem 8.33: Accuracy of $P_k(\Omega^h)$

Let $\{\Omega^h\}$ for $0 < h \leq 1$ be a non-degenerate family of simplex meshes of a polyhedral domain, $\Omega \subset \mathbb{R}^n$. Let $\mathcal{V}^h = P_k(\Omega^h)$ with $k + 1 - n/2 > 0$ and a suitable choice of nodes for the degrees of freedom of $P_k(\Omega^h)$. Let I^h be such that $I^h w \in \mathcal{V}^h$ is the interpolant of $w \in C^0(\Omega)$. Then, there exists a constant, C, depending on the choice of nodes, n, k, and ρ such that if $u \in H^{k+1}(\Omega)$, then

$$\left(\sum_{\tau \in \Omega^h} \|u - I^h u\|_s^2 \right)^{\frac{1}{2}} \leq C h^{k+1-s} |u|_{k+1}, \tag{8.70}$$

for $0 \leq s \leq k + 1$.

Types of Elements and Approximation Properties

Triangular meshes on conforming polynomial spaces

Theorem 8.33: Accuracy of $P_k(\Omega^h)$

Let $\mathcal{V}^h = P_k(\Omega^h)$ *with* $k + 1 - n/2 > 0$ *and a suitable choice of nodes for the degrees of freedom of* $P_k(\Omega^h)$. *Let* I^h *be such that* $I^h w \in \mathcal{V}^h$ *is the interpolant of* $w \in C^0(\Omega)$. *Then, there exists a constant,* C, *depending on the choice of nodes,* n, k, *and* ρ *such that if* $u \in H^{k+1}(\Omega)$ *then*

$$||u - I^h u||_s \leq C h^{k+1-s} |u|_{k+1} \tag{8.70}$$

for $0 \leq s \leq k + 1$.

Types of Elements and Approximation Properties

Triangular meshes on conforming polynomial spaces

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Let $\mathcal{V}^h = P_k(\Omega^h)$ *with* $k+1-n/2 > 0$ *and a suitable choice of nodes for the degrees of freedom of* $P_k(\Omega^h)$. *Let* I^h *be such that* $I^h w \in \mathcal{V}^h$ *is the interpolant of* $w \in C^0(\Omega)$. *Then, there exists a constant,* C*, depending on the choice of nodes,* n*,* k*, and* ρ *such that if* $u \in H^{k+1}(\Omega)$*, then*

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for $0 \leq s \leq k+1$.

$$I^h u = u^h$$

Types of Elements and Approximation Properties

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Let $\mathcal{V}^h = P_k(\Omega^h)$ with $k + 1 - n/2 > 0$ and a suitable choice of nodes for the degrees of freedom of $P_k(\Omega^h)$. Let I^h be such that $I^h w \in \mathcal{V}^h$ is the interpolant of $w \in C^0(\Omega)$. Then, there exists a constant, C, depending on the choice of nodes, n, k, and ρ such that if $u \in H^{k+1}(\Omega)$, then

$$||u - I^h u||_s \leq C h^{k+1-s} |u|_{k+1} \tag{8.70}$$

for $0 \leq s \leq k+1$.

$$||u||_s = \sum_{\alpha \leq s} ||D^\alpha u||_2$$

L^2-based Sobolev norm

$$|u|_p = \left(\int_\Omega |u|^p \, dx \right)^{1/p}$$

L^p-norm

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Triangular meshes on conforming polynomial spaces

Theorem 8.33: Accuracy of $P_k(\Omega^h)$

Let $\mathcal{V}^h = P_k(\Omega^h)$ with $k + 1 - n/2 > 0$ and a suitable choice of nodes for the degrees of freedom of $P_k(\Omega^h)$. Let I^h be such that $I^h w \in \mathcal{V}^h$ is the interpolant of $w \in C^0(\Omega)$. Then, there exists a constant, C, depending on the choice of nodes, n, k, and ρ such that if $u \in H^{k+1}(\Omega)$, then

$$||u - I^h u||_s \leq C h^{k+1-s} |u|_{k+1} \tag{8.70}$$

for $0 \leq s \leq k+1$.

Example: $\mathcal{V}^h = P_1(\Omega^h)$

$$u \in H^2(\Omega)$$

$$0 \leq s \leq 2$$

$$s = 0: \ \|u - I^h u\|_0 \leq C h^2 |u|_2,$$

$$s = 1: \ \|u - I^h u\|_1 \leq C h |u|_2.$$

Types of Elements and Approximation Properties

Triangular meshes on conforming polynomial spaces

Accuracy of $P_k(\Omega^h)$ **:** $||u - I^h u||_s \leq Ch^{k+1-s}|u|_{k+1}$

Example: $\mathcal{V}^h = P_1(\Omega^h)$

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$s = 0: \ \|u - I^h u\|_0 \leq Ch^2|u|_2,$

$s = 1: \ \|u - I^h u\|_1 \leq Ch|u|_2.$

$\mathcal{V}^h = P_2(\Omega^h)$

$u \in H^3(\Omega)$

$0 \leq s \leq 3$

$s = 0: \ \|u - I^h u\|_0 \leq Ch^3|u|_3,$

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Types of Elements and Approximation Properties

Combine Cea's Lemma with the accuracy of $P_k(\Omega^h)$ estimate

Cea's Lemma for $P_k(\Omega^h)$: $||u - u^h||_k \leq \frac{c_1}{c_0} \min_{u_h \in P_k(\Omega^h)} ||u - u_h||_k$

$$\mathcal{V}^H = P_1(\Omega^h)$$

$$u \in H^2(\Omega)$$

$$0 \leq s \leq 2$$

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$$\|u - u^h\|_1 \underbrace{\leq \frac{c_1}{c_0} \min_{v^h \in P_1(\Omega^h)} \|u - v^h\|_1}_{\text{Céa's lemma}} \underbrace{\leq Ch|u|_2}_{\text{Theorem 8.33}}$$

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Aubin-Nitsche duality argument: (Theorem 4.9) $\|u - u^h\|_0 \leq Ch^2|u|_2$

Types of Elements and Approximation Properties

Combine Cea's Lemma with the accuracy of $P_k(\Omega^h)$ estimate

Let's verify that we get the expected accuracy in practice

$$-\nabla \cdot \nabla u = f(x) \text{ for } x \in \Omega = [-1, 1]^2,$$
$$u = 0 \text{ on } \partial\Omega.$$

Choosing a test problem, where the solution is of the form

$$u(\boldsymbol{x}) = \sin\left(\pi \frac{x-1}{2}\right)\cos(\pi y),$$

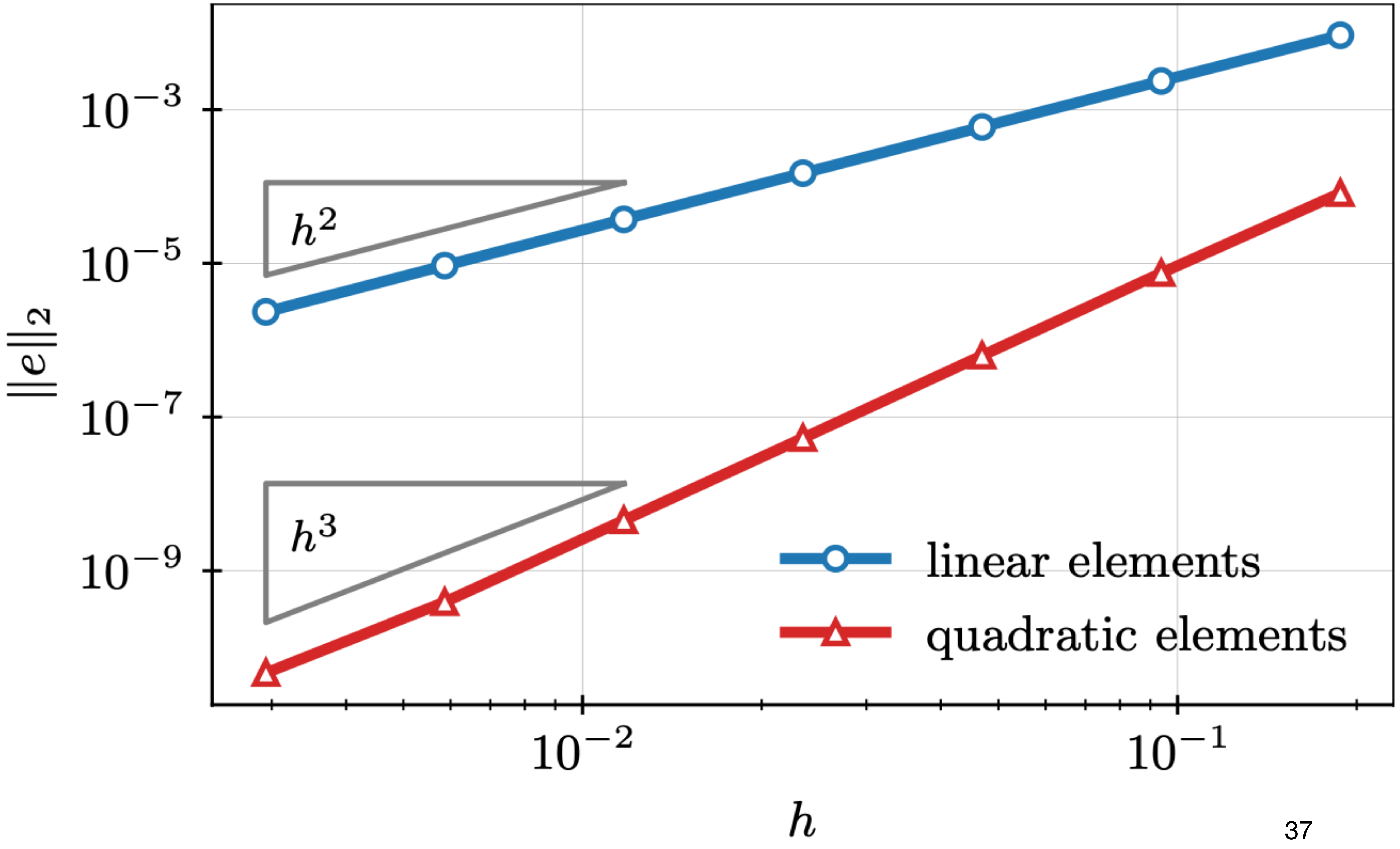
yields a forcing function,

$$f(\boldsymbol{x}) = \frac{5}{4}\pi^2 \sin\left(\pi \frac{x-1}{2}\right)\cos(\pi y).$$

Types of Elements and Approximation Properties

Combine Cea's Lemma with the accuracy of $P_k(\Omega^h)$ estimate

Let's verify that we get the expected accuracy in practice



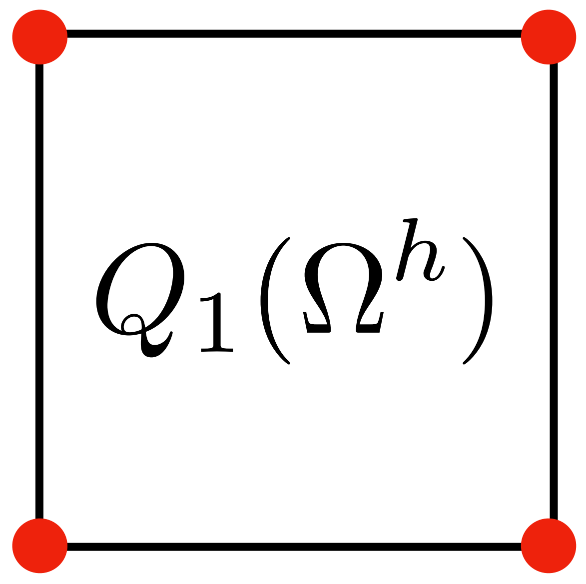
$$\|u - u^h\|_0 \leq Ch^2|u|_2 \text{ for } P_1(\Omega^h) \text{ elements}$$

$$\|u - u^h\|_0 \leq Ch^3|u|_3 \text{ for } P_2(\Omega^h) \text{ elements}$$

Types of Elements and Approximation Properties

Quadrilateral meshes on conforming polynomial spaces

$$Q_k(\Omega^h) = \Big\{v \in C^0(\Omega^h) \Big| \forall \tau \in \Omega^h, v(\boldsymbol{x}) \text{ is a polynomial with possible terms } x^\alpha y^\beta \text{ for } \max(\alpha, \beta) \leq k \text{ on } \tau\Big\}$$



$$\phi_i(x, y) = a_i + b_i x + c_i y + d_i x_i y_i$$

$$\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$