CS555

(same show, different person)

Andreas Kloeckner

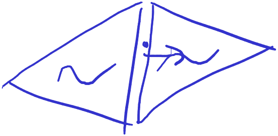
Today:

- ecological niche
- setup
- stability proof
- implementing

| high accuracy (smooth)
| complicated geo
| time-domain

flux

$$u_t + f(u)_x = 0$$



discontinuous Galerkin (dG)

Outline

Outline

Conservation laws

Goal: Solve *conservation laws* on bounded domain $\Omega \subset \mathbb{R}^n$:

$$\boldsymbol{q}_t + \nabla \cdot \boldsymbol{F}(\boldsymbol{q}) = 0$$

Example: Maxwell's Equations

$$\partial_t \boldsymbol{D} - \nabla \times \boldsymbol{H} = -\boldsymbol{j}, \qquad \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0,$$
$$\nabla \cdot \boldsymbol{D} = \rho, \qquad \nabla \cdot \boldsymbol{B} = 0.$$

What do we do with the divergence constraints?

Ignore divergence constraints

Rewriting Maxwell's

Let $\boldsymbol{q} = (D_x, D_y, D_z, B_x, B_y, B_z)^T$. Consider $\boldsymbol{D} = \epsilon \boldsymbol{E}$ and $\boldsymbol{B} = \mu \boldsymbol{H}$.

$$\partial_t \boldsymbol{D} - \nabla \times \boldsymbol{H} = -0, \qquad \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0.$$

Assume ϵ, μ constant. Rewrite in conservation law form: $\boldsymbol{q}_t + \nabla \cdot F(\boldsymbol{q}) = 0$

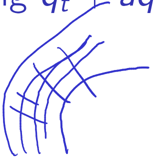
$$\vec{q} = \begin{pmatrix} \vec{D} \\ \vec{B} \end{pmatrix} \qquad q_t + \nabla \cdot \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & -\frac{B_x}{\epsilon} & -B_y \\ B_z/\epsilon & 0 & -\frac{B_x}{\epsilon} \\ -\frac{B_y}{\epsilon} & \frac{B_x}{\epsilon} & 0 \\ & \ddots & \end{pmatrix} \end{array} = 0$$

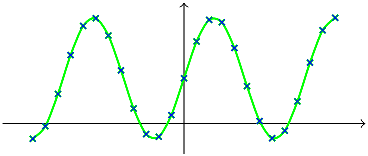
components

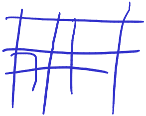
Could we also define $\boldsymbol{q} = (E_x, E_y, E_z, H_x, H_y, H_z)^T$?

Outline

Solving $q_t + aq_x = 0$: Finite Differences







⊕ Simple to implement

⊕ High order

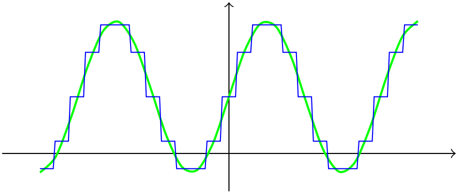
no complex geometry

⊕ theory available

$$D_t^- + aD_x^- = 0$$

$$D_t^+ f := \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Solving $q_t + aq_x = 0$: Finite Volume



⊕ robust for nonlinear, w/ theory

~ unrealistic for complex geometry

⊕ explicit in time

$$\bar{q}_k := \int_{(k-1/2)\Delta x}^{(k+1/2)\Delta x} q(x)dx$$

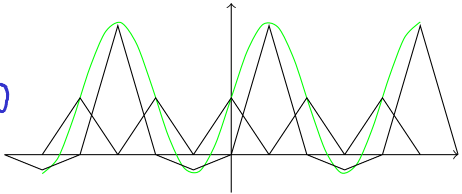
$$\Delta x \partial_t \bar{q}_k + f^{k+1/2} - f^{k-1/2} = 0$$

$f^{k\pm1/2}$: flux "reconstructions"

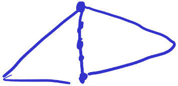
Solving $q_t + aq_x = 0$: Finite Elements

$\partial_t q + \nabla \cdot F(q) = 0$

$M\bar{q}_t + D \ldots$



shared in CG



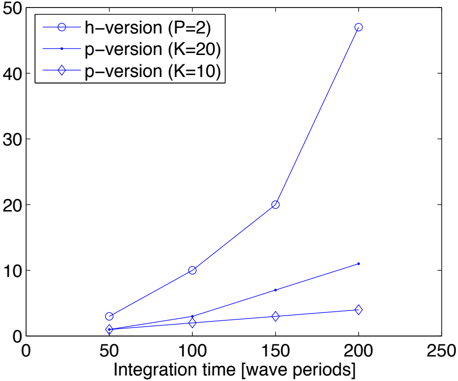
$$\int_\Omega q_t^N \phi + a q_x^N \phi \, dx = 0$$

for ϕ in a test space.

- ⊕ high-order
- ⊕ geom. flexibility
- ⊕ solid theory for elliptic
- ⊖ inherently implicit / have to solve
- ⊖ theory not super solid in hyperbolic

" continuous Galerkin

Do we really want high order?



Time to compute solution at 5% error

Big assumption?

Smooth

Figure from talk by Jan Hesthaven

Summarizing

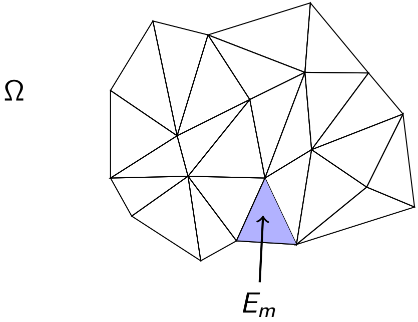
Want flexibility of finite elements *without* the drawbacks.

nD derivation

1D at points

Outline

Developing the Scheme

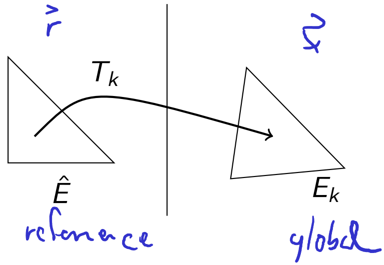


What do do about unbounded domains?

- PML "perfectly matched layers"
- absorbing BCs

Dealing with the Mesh, Part I

For each cell E_k, find a ref-to-global map T_k:



$$T_k : \hat{E} \to E_k$$

$$\boldsymbol{x} = (x, y, z) = T_k(r, s, t) = T_k(\boldsymbol{r})$$

- T_k affine for straight-sided simplices: $T_k(\boldsymbol{r}) = A\boldsymbol{r} + \boldsymbol{b}$
- Curved elements also possible: iso/sub/super-parametric

Dealing with the Mesh, Part II

Based on knowledge of how to do this on $\hat{E}$:

Can now *integrate* on Ω:

$$\int_{\Omega} f\,dx = \sum_{E_k} \int f\,dx = \sum_{E_k} \int_{\hat{E}} f \left| \frac{dx}{dr} \right| dr$$

and *differentiate* on Ω:

$$\frac{df}{dx} = \underbrace{\frac{dr}{dx}}\cdot\underbrace{\frac{df}{dr}}_{\text{ref element quantity}}$$

Jacobian of T_k^{-1}?

$$\frac{dx}{dr}\cdot\frac{dr}{dx} = \mathrm{Id} \iff \frac{dr}{dx} = \left(\frac{dx}{dr}\right)^{-1}$$

Dealing with the Mesh, Part III

Approximation basis set on E_k?

$$\varphi_i^k(\vec{x}) = \varphi_i(T_k^{-1}(\vec{x}))$$

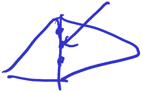
What function space do we get if T_k is non-affine?

Approximation results nontrivial

Going Galerkin

$$\int_{E_k} q_t^k \phi + (\nabla \cdot F^k)\phi dx = 0$$

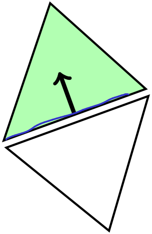
cG:



Integrate by parts:

$$0 = \underbrace{\int_{E_k} q_t^k \varphi \, dx}_{\text{cell-local}} - \underbrace{\int_{E_k} F^k \cdot \nabla \varphi \, dx}_{\text{cell-local}} + \int_{\partial E_k} (F^k \cdot \hat{n})^* \varphi \, dS_x$$

Mass matrix is block-diagonal



Problem?

$(F^k \cdot n)^*$ num. flux, involves both sides of the el. interface.

Strong-Form DG

$\frac{h}{N^2}$

Weak form:

$$0 = \int_{E_k} q_t^k \phi dx - \int_{E_k} F^k \cdot \nabla \phi dx + \int_{\partial E_k} (F^k \cdot \hat{\boldsymbol{n}})^* \phi dx$$

Integrate by parts *again*:

$$0 = \int q_t^h \varphi + (\nabla \cdot F_u) \varphi) dx + \int_{\partial E_k} (F_u \cdot n_u)^* - (F_u \cdot \hat{n})^- dS_x$$

"local"

Outline

Accuracy and Stabillity

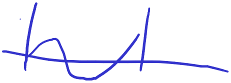
In DG: what provides accuracy? what provides stability?

local parts : provides accuracy
Flux : stability

Following slides based on material by Tim Warburton

Stability: Basic Setup (1/2)

L^2 stability



$\partial_t \|q\|_{L^2}^2 \leq 0$

$$0 = \int_{E_k} q_t^k \phi dx - \int_{E_k} F^k \cdot \nabla \phi dx + \int_{\partial E_k} (F^k \cdot \hat{\boldsymbol{n}}) \phi dS_x$$

$\partial_t (q^2)$

$F(q) = (aq, 0, 0)$. Choose $\varphi = q_k$.

$$0 = \int q_t^k q_k dx - \int_{E_k} a q_k \vec{e}_x \cdot \nabla q_k dx + \int_{\partial E_k} (a q_k \vec{e}_x \cdot \hat{n})^* q_k dS_x$$

$$0 = \int q_t^k q_k dx - \int_{E_k} a q_k (\partial_x q_k) dv + \int_{\partial E_k} (a q_k \vec{e}_x \cdot \hat{n})^* q_k dS_x$$

$$0 = \frac{\partial_t}{2} \int_{E_k} q_k^2 dx - \int_{E_k} a q_k (\partial_x q_k) dv + \int_{\partial E_k} (a q_k \vec{e}_x \cdot \hat{n})^* q_k dS_x$$

$$0 = \frac{\partial_t}{2} \|q_k\|_{L^2(E_k)} - \int_{E_k} a q_k (\partial_x q_k) dv + \int_{\partial E_k} (a q_k \vec{e}_x \cdot \hat{n})^* q_k dS_x$$

$\overset{!}{\leq} 0.$

Stability: Basic Setup (2/2)

$$\frac{\partial_t \|q_k\|_{2,E_k}^2}{2} = \int_{E_k} a q_k \partial_x q_k dx - \int_{\partial E_k} (a q_k n_x)^* q_k dS_x$$

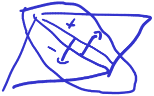
$$\int f \partial_x f = -\int f \partial_x f + \int_\partial f^2 dx$$

$$\int f \partial_x f = \frac{1}{2} \int_\partial f^2 dx$$

$\Rightarrow$

$$\frac{\partial_t \|q_k\|_{2,E_k}^2}{2} = \int_{\partial E_k} \frac{a (q_k^-)^2 n_x}{2} - (a q_k n_x)^* q_k dS_x \overset{!}{\leq} 0$$

Stability: Going Global



$$\frac{\partial_t \|q_k\|^2_{2,E_k}}{2} = \int_{\partial E_k} \frac{a(q_k)^2 n_x}{2} - (a q_k n_x)^* q_k \, dS_x$$

$$\frac{\partial_t \|q_k\|^2_{2,\Omega}}{2} = \sum_k \int_{\partial E_k} \frac{a(q_k^-)^2 n_x}{2} - (a q_k n_x)^* q_k \, dS_x$$

$$= \sum_{f \in \text{faces}} \int_f \frac{a(q_k^+)^2 n_x^+}{2} - (a q_k n_x^+)^* q_k$$

assumed same.

$$- \frac{a(q_k^-)^2 n_x^+}{2} + (a q_k n_x^+)^* q_k^- \, dS_k$$

("boundaries")
assumption

Gather up

$$\frac{\partial_t \|q_k\|^2_{2,\Omega}}{2} = \sum_{f \in \text{faces}} \Bigg(\int_f \frac{a(q_k^+)^2 n_x^+}{2} - (aq_k n_x)^*_{\oplus} q_k^+ \, dS_x$$

▷ assume a is constant

$$+ \int_f \frac{a(q_k^-)^2 n_x^-}{2} - (aq_k n_x)^*_{\ominus} q_k^- \, dS_x \Bigg) \stackrel{?}{\le} 0$$

same

▷ Neglected domain bdry

$$\partial_t \|q_k\|_{L^2,\Omega} = \sum_{f \in \text{faces}} \underbrace{\int_f a n_x \frac{(q_k^-)^2 - (q_k^+)^2}{2} - (a q_k n_x)^* (q_k^- - q_k^+) \, dS_x}$$

$$= \sum_f \underbrace{\int \left(a n_x \left(\frac{q_k^- + q_k^+}{2} \right) - (a q n_x)^* \right) \cdot \left[q_k^- - q_k^+ \right] dS_x}_{\le 0.} \le 0$$

Picking a Flux

Want:

$$(*) = \left(a n_x^- \frac{q_k^- + q_k^+}{2} - (a q_k n_x)^*_- \right) (q_k^- - q_k^+) \overset{!}{\le} 0$$

Ideas?

One choice:

$$(a q_k n_x)^* := a n_x \frac{q_k^+ + q_k^-}{2}$$

central flux

$$\partial_t \|q_k\|_{L^2, \Omega} = 0$$

Picking a flux, attempt two

Want:

$$(*) = \left(a n_x^- \frac{q_k^- + q_k^+}{2} - (a q_k n_x)^*_- \right) (q_k^- - q_k^+) \overset{!}{\le} 0$$

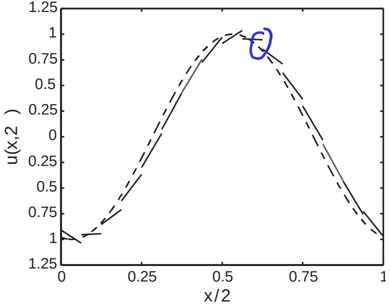
More ideas?

$$(a\, q_k n_x)^* := a n_x \frac{q_k^+ + q_k^-}{2} + \alpha \frac{q_k^- - q_k^+}{2}$$

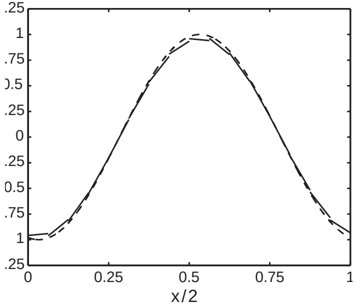
local Lax-Friedrichs

Comparing Fluxes (1/3)

Central



Upwind



Upwind penalizes jumps!

Figure from talk by Jan Hesthaven

Comparing Fluxes (2/3)

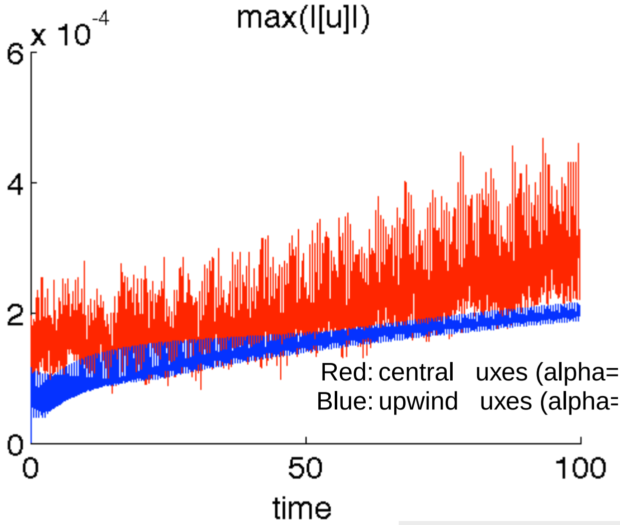


Figure from lecture by Tim Warburton